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A novel long-range spin chain and planar $\mathcal{N}=4$ super Yang-Mills

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Abstract: We probe the long-range spin chain approach to planar $\mathcal{N}=4$ gauge theory at high loop order. A recently employed hyperbolic spin chain invented by Inozemtsev is suitable for the $\mathfrak{su}(2)$ subsector of the state space up to three loops, but ceases to exhibit the conjectured thermodynamic scaling properties at higher orders. We indicate how this may be bypassed while nevertheless preserving integrability, and suggest the corresponding all-loop asymptotic Bethe ansatz. We also propose the local part of the all-loop gauge transfer matrix, leading to conjectures for the asymptotically exact formulae for all local commuting charges. The ansatz is finally shown to be related to a standard inhomogeneous spin chain. A comparison of our ansatz to semi-classical string theory uncovers a detailed, non-perturbative agreement between the corresponding expressions for the infinite tower of local charge densities. However, the respective Bethe equations differ slightly, and we end by refining and elaborating a previously proposed possible explanation for this disagreement.

Keywords: AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

Contents

1. Introduction

1.1 Spins ...

The calculation of anomalous dimensions of local composite operators in a conformal quantum field theory such as $\mathcal{N}=4$ gauge theory in four dimensions is difficult even in perturbation theory. At one loop the relevant Feynman diagrams are easily computed, but in general a formidable mixing problem for fields of equal classical dimension has to be resolved. At higher loops the mixing problem worsens, and, in addition, the Feynman diagram technique rapidly becomes prohibitive in complexity. Recently much progress was achieved in dealing with both problems. It was understood, initially for a scalar subsector of the fields, that the computation can be quite generally reformulated in a combinatorial fashion [1, 2, 3, 4], and that this combinatorics may be efficiently treated by hamiltonian methods [5, 6].

At one loop, Minahan and Zarembo recognized that this hamiltonian is, in the *planar* limit, identical to the one of an *integrable* quantum spin chain [5]. This picture was successfully extended to all $\mathcal{N}=4$ fields in [7], exploiting the planar structure of the complete non-planar one-loop dilatation operator obtained in [8]. The resulting (non-compact) $\mathfrak{psu}(2,2|4)$ super spin chain does not only extend the integrable structures of [5], it also unifies them with the ones observed earlier for different types of operators in the context of QCD [9, 10, 11].

The second development was the realization that hamiltonian methods are also applicable at higher loops [12].[1] Most importantly, it was shown that planar *integrability* extends to at least two loops, and it was conjectured that the full non-perturbative planar dilatation operator of $\mathcal{N}=4$ theory is identical to the hamiltonian of some integrable *long-range* spin chain. This was achieved by studying the two-loop deformations of the hidden commuting charges responsible for the one-loop integrability. Based on this (and certain further assumptions, see below) the planar three-loop dilatation operator for the $\mathfrak{su}(2)$ subsector of the state space was derived. One of its predictions was the previously unknown three-loop anomalous dimension of the Konishi field. Very recently, entirely independent arguments resulted in a conjecture for the three-loop anomalous dimension of a twist-two Konishi descendant [13].[2] It is based on extracting the $\mathcal{N}=4$ anomalous dimensions of twist operators from the exact QCD result by truncating to contributions of the 'highest transcendentality'. The three-loop QCD result recently became available after

[1] One should not confuse the hamiltonian approach with the integrable spin chain approach. E.g. in [12] the full non-planar one- and two-loop dilatation operator was derived in the $\mathfrak{su}(2)$ sector. It acts on a grand-canonical ensemble of disconnected spin chains. The $\frac{1}{N}$ corrections lead to spin chain splitting and joining, as in [6], and break integrability.

[2] This conjecture has been confirmed by an explicit calculation in $\mathcal{N}=4$ at the two-loop level.

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an impressive, full-fledged and rigorous field theoretic computation by Moch, Vermaseren and Vogt [14]. The conjecture of [13] *agrees* with the prediction of [12] in a spectacular fashion.

What is the evidence that the full $\mathcal{N} = 4$ planar dilatation operator is indeed described by an integrable long-range spin chain? In [15] three-loop integrability was *proven* for the maximally compact closed $\mathfrak{su}(2|3)$ subsector of $\mathfrak{psu}(2,2|4)$. Independent confirmation comes from a related study of dimensionally reduced $\mathcal{N} = 4$ theory at three loops [16]. Furthermore, the procedure of [12] of deriving the $\mathfrak{su}(2)$ dilatation operator by assuming integrability was pushed to four loops, leading to a unique result after imposition of two further assumptions [17]. The first of these is suggested by rather firmly established structural constraints derived from inspection (as opposed to calculation) of Feynman diagrams. The second, somewhat less compelling, assumption postulates a certain *thermodynamic scaling behavior*, i.e. the L dependence of the anomalous dimension in the limit of large spin chain length L.

What is the relevance of the observed integrability? Apart from its ill-understood conceptual importance for planar $\mathcal{N} = 4$ theory, it allows for very efficient calculations of anomalous dimensions by means of the Bethe ansatz, as first derived at one loop in [5, 7]. For "long" composite operators where $L \gg 1$ this computational tool is not only useful, but indispensable. Beyond one loop, a Bethe ansatz is also expected to exist, as the latter is based on the principle of factorized scattering. This means that the problem of diagonalizing a spin chain with M excitations (magnons) may be reduced to the consideration of a sequence of pairwise interactions, i.e. the two-body problem. This principle is one of several possible ways to characterize integrability. And indeed a Bethe ansatz technique was derived in [18] for the $\mathfrak{su}(2)$ sector up to three-loops. This involved embedding the three-loop dilatation operator of [12] into an integrable long-range spin chain invented by Inozemtsev [19, 20].

At four loops, however, the Inozemtsev chain clashes with the postulate of thermodynamic scaling in an irreparable fashion [18]. It thus also contradicts the four-loop integrable structure found in [17]. This is somewhat surprising, as Inozemtsev presented arguments which suggest that his integrable long-range chain should be the most general one possible. One might wonder whether a spin chain with "good" thermodynamic behavior could lead to inconsistencies at even higher loop levels. In section 2 we will see that this does not happen up to *five* loops. In fact, it appears that the principles of integrability, field theory structure, and thermodynamic behavior result in a *unique*, novel long-range spin chain. As a crucial second test of this claim, we will show, by working out a large number of rather non-trivial examples, that the scattering of our new chain is still factorized up to five loops. A byproduct is the successful test of the validity of the three-loop ansatz of [18] for a larger set of multi-magnon states. Our study allows us to find the Bethe ansatz corresponding to the new chain. What is more, our findings suggest a general pattern for the scattering which appears to be applicable at arbitrary order in perturbation theory. Stated differently, we propose an integrability-based non-perturbative procedure for calculating anomalous dimensions in the $\mathcal{N} = 4$ model without the need of knowing the precise all-loop structure of its dilatation operator!

Of course the reader should keep in mind that our model is merely an *ansatz* for the treatment of the gauge theory, and proving (or disproving) it will require new insights.[3] Furthermore, the validity of the all-loop Bethe ansatz we are proposing is still subject to one serious restriction. It is, as in [20, 18], *asymptotic* in the sense that the length of the chain (and thus of the operator) is assumed to exceed the range of the interaction (and thus the order in perturbation theory). We hope that this restriction will be overcome in the future, as it might then allow to find the spectrum of planar $\mathcal{N}=4$ of finite length operators. In addition, we suspect the restriction to be at the heart of a recently discovered vexing discrepancy between the anomalous dimensions of certain long operators and the energies of the related IIB super string states in the $AdS_5 \times S^5$ background.

1.2 ... and strings

While anomalous dimensions are of intrinsic interest to the gauge field theory, further strong motivation for their study comes from a conjectured relation to energies of superstrings on curved backgrounds. Spin chains in their ferromagnetic ground state describe $\mathcal{N}=4$ half-BPS operators. Long spin chains near the ground state, with a finite number of excitations, correspond then to near-BPS operators. These are dual, via the AdS/CFT correspondence, to certain string states carrying large charges, whose spectrum can be computed exactly as they are effectively propagating in a near-flat metric [21]. This yields an all-loop prediction of the anomalous dimensions of the near-BPS operators. The prediction is interesting even qualitatively, as it leads to the above mentioned thermodynamic "BMN" scaling behavior of the spin chain. One then finds, combining this behavior with integrability, that the BMN prediction is also quantitatively reproduced, as will be argued below up to five loops. This feature is a strong argument in favor of our new chain, and thus against (beyond three loops) the Inozemtsev model. The all-loop extension of the five-loop spin chain assumes the repetition of the observed pattern ad infinitum, and therefore reproduces the quantitative BMN formula to all orders by construction. One should nevertheless stress that BMN scaling is very hard to *prove* on the gauge side (much harder than e.g. integrability!). So far this has not been done rigorously beyond two loops. Therefore the Inozemtsev model has not yet been completely ruled out either.[4]

The BMN limit is not the only situation where a quantitative comparison between long gauge operator dimensions and large charge string energies was successful. The large charge limit may be interpreted as a semi-classical approximation to the string sigma model [22]. Following an idea of Frolov and Tseytlin, this allowed to perform explicit calculations of the energies of strings rapidly spinning in two directions on the five sphere [23, 24, 25] and successfully comparing them, at one-loop, to $\mathfrak{su}(2)$ Bethe ansatz computations [26, 27]. Using the mentioned higher-loop Inozemtsev-Bethe ansatz, it was possible to also confirm

[3]Accordingly, when working out the potential consequences of our ansatz for the gauge theory below, we will for simplicity mostly write "gauge theory result" instead of "gauge theory result under the assumption of the validity of the novel long-range spin chain ansatz", etc.

[4]The result of [14, 13] fixes one of the two remaining [15] parameters of the three-loop dilatation operator of [12] which are left undetermined unless one assumes BMN scaling behavior. Likewise, one further three-loop dimension for a different field in the $\mathfrak{su}(2)$ sector would complete the proof of scaling at this order.

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the matching at two loops [18]. The string sigma model is classically integrable and this makes explicit computations of the energies and charges feasible [25, 28]. The agreement accordingly also extends to the tower of one- and two-loop commuting charges [29, 30]. A more intuitive understanding of this agreement in sigma model language was achieved in [31, 32].

Unfortunately this encouraging pattern breaks down at *three* loops [18]. A similar three-loop disagreement appeared earlier in the string analysis of the near BMN limit presented in [33, 34]. This case will be discussed in detail in section 2.6. One might wonder whether the trouble is either due to a faulty embedding of the three-loop dilatation operator into the Inozemtsev long-range spin chain, or else, due to problems with the three-loop asymptotic Inozemtsev-Bethe ansatz. In this paper an extended and detailed study of the multi-magnon diagonalization using this ansatz definitely rules out these potential pitfalls. Let us stress once more that the three-loop disagreement of the Inozemtsev-Bethe ansatz with string theory is *unrelated* to the four-loop breakdown of thermodynamic scaling in the Inozemtsev model. In this paper we are definitely bypassing the second problem, and suggesting a potential explanation of the first problem in section 4. There we will refine the conjecture, first made in [18], that the disagreement might be explained as an order of limits problem. In particular, we shall argue in the last section of this paper that if we were to implement the same scaling procedure as in string theory, we should include *wrapping interactions* into the gauge theory computations. These are precisely excluded by the weak-coupling asymptotic Bethe ansatz. The refinement also allows to gain a qualitative understanding why the strict BMN limit might indeed agree at all orders, while subtleties arise in the near BMN and the Frolov-Tseytlin situations.

Further confirmation for this picture, as well as for the validity of our novel long-range spin chain, comes from a detailed all-loop comparison of the long-range asymptotic Bethe ansatz with the classical Bethe equation for the string sigma model. The latter was recently derived for the $\mathfrak{su}(2)$ sector in an important work by Kazakov, Marshakov, Minahan and Zarembo [35]. It allowed to successfully compare the string and gauge integrable structures[5] by showing that the classical equation may be mapped to the thermodynamic limit of the quantum spin chain Bethe and Inozemtsev-Bethe equations at, respectively, one and two-loops [35]. Here we will extend this comparison in section 3 to all-loops. We find the intriguing result that the *local excitations* of string and gauge theory agree to all orders in perturbation theory! The Bethe equations describing the dynamics of the excitations however differ, leading to differences in the expectation values of the global charges. We suspect that this is due to the global effect of wrapping interactions, as will be discussed in the final section 4. These disagreements between the energy eigenvalues of the string sigma model on the one hand and those of the long-range spin chain (in the thermodynamic limit) on the other hand will be illustrated by way of example in appendix C. There we explicitly derive the energies of the folded and the circular string using the Bethe ansätze for both the novel spin chain and the string sigma model as introduced in [35].

[5]This was shown earlier on two specific examples by using the Bäcklund transformation [29] (see also [30]). One major advantage of the systematic approach of [35] is its generality. It would be interesting to also treat the Bäcklund approach in a general fashion.

Let us end by mentioning a crucial issue which is not addressed in this paper, namely the extension to subsectors larger than the closed $\mathfrak{su}(2)$ spin $\frac{1}{2}$ chain. In fact, essentially nothing is known about the Bethe ansatz for larger sectors, except at one loop [5, 7]. At this loop order, there is much evidence that the triality between string theory, gauge theory and spin chains extends to other sectors, see e.g. the string [36]–[44] and Bethe [45, 46, 47] computations, or even to other superconformal models containing open strings [48, 49]. See also the review paper [50]. Subsectors containing covariant derivatives or field strengths should be very important in the QCD context [9, 10, 11], cf. also the very recent paper [51]. It would be extremely interesting to find the analog of our asymptotic Bethe ansatz for some larger closed sector, and ideally for the full $\mathfrak{psu}(2,2|4)$ super spin chain. Given that the full super spin chain is certainly *dynamic* [15], i.e. the length of the chain becomes itself a quantum variable beyond one loop, such an ansatz, if it exists at all, will presumably contain novel features not yet encountered in traditional exactly solvable spin chains. In particular it should be fascinating to see how such an ansatz might reconcile two seemingly contradictory features of the interactions of the elementary excitations of such chains. These features are, for one, the principle of elastic scattering, as usually required by integrability, and, secondly, the occurrence of particle production in dynamic, long-range spin chains, where fermionic and bosonic degrees of freedom are not separately conserved. Could it be that supersymmetry will lead to a generalization of the traditional notion of a factorized S-matrix?

2. The long-range spin chain

The $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ SYM consists of local operators composed from two charged scalar fields $\mathcal{Z}, \phi$. In the planar limit, local operators are single trace, and of the form

$$\operatorname{Tr}\left(\mathcal{Z}^{L-M}\phi^{M}\right)+\cdots\,, \tag{2.1}$$

where the dots indicate linear mixing of the elementary fields $\mathcal{Z}, \phi$ inside the trace in order to form eigenstates of the dilatation generator. These can be represented by eigenstates of a cyclic $\mathfrak{su}(2)$ quantum spin chain of length L with elementary spin $\frac{1}{2}$ [5]. In this picture M is the number of "down spins" or "magnons", and the dilatation operator, which closes on this subsector [12], corresponds to the spin chain hamiltonian. For future use let us also introduce the letter J for the number of "up" spins

$$J = L - M\,, \tag{2.2}$$

which is the standard notation [21] for the total $\mathfrak{so}(2) \subset \mathfrak{so}(6)$ charge of the chiral scalar fields $\mathcal{Z}$. Minahan and Zarembo have discovered that the dilatation operator at the one-loop level is in fact integrable [5] and thus isomorphic to the Heisenberg $\mathrm{XXX}_{1/2}$ spin chain. In [12] it was shown that integrability extends to two-loops and conjectured that it might hold at all orders in perturbation theory or even for finite 't Hooft coupling constant.

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In this section we will investigate a possible extension of the dilatation operator to higher loop orders. We will make use of three key assumptions:

(i) Integrability,

(ii) field theoretic considerations, and

(iii) BMN scaling behavior.

These are not firm facts from gauge theory, but there are reasons to believe in them. For example, at three-loops integrability follows from (firm) field theoretic constraints and superconformal symmetry [15]. One could argue that BMN scaling is an analog of the 'highest transcendentality' conjecture (see [13] and references therein): in $\mathcal{N}=4$ SYM all contributions scale with the maximum allowed power of $1/J$. What is more, a conjecture for a three-loop anomalous dimension in $\mathcal{N}=4$ SYM [13], which rests on entirely unrelated assumptions while being based on a rigorous *tour-de-force* computation for QCD [14], agrees with the prediction of the spin chain and thus confirms our premises to some extent. Whether or not the assumptions are fully justified in (perturbative) $\mathcal{N}=4$ SYM will not be the subject of this section, but we believe that the model shares several features with higher-loop gauge theory and therefore deserves an investigation. Intriguingly, it will turn out to be *unique* up to (at least) five-loops and agree with the excitation energy formula in the BMN limit! At any rate, this makes it a very interesting model to consider in its own right. For a related, very recent study see [52].

There are two approaches to integrable quantum spin chain models. One uses the hamiltonian and the corresponding commuting charges. The other employs factorized scattering and the Bethe ansatz technique. We will discuss these two approaches in the following two subsections. By means of example we shall demonstrate the equivalence of both models in section 2.5 and appendix A.

2.1 Commuting charges

Let us start by describing the set of charges as operators acting on the spin chain. Introducing a coupling constant g by

$$g^2 = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2} = \frac{\lambda}{8\pi^2}\,, \tag{2.3}$$

we expand the charges in a Taylor series

$$\mathbf{Q}_r(g) = \sum_{\ell=1}^{\infty} \mathbf{Q}_{r,2\ell-2}\, g^{2\ell-2}\,. \tag{2.4}$$

The dilatation operator $\mathbf{D}$ can be expressed in terms of the spin chain hamiltonian $\mathbf{H}$ which is defined as the second charge $\mathbf{Q}_2$

$$\mathbf{D}(g) = L + g^2\,\mathbf{H}(g), \qquad \mathbf{H}(g) = \mathbf{Q}_2(g)\,. \tag{2.5}$$

Any $\mathfrak{su}(2)$ invariant interaction can be written as a permutation of spins. These can in turn be represented in terms of elementary permutations $\mathcal{P}_{p,p+1}$ of adjacent fields. A generic term will thus be written as[6]

$$\{p_1, p_2, \ldots\} = \sum_{p=1}^{L} \mathcal{P}_{p+p_1,p+p_1+1}\mathcal{P}_{p+p_2,p+p_2+1}\cdots . \tag{2.6}$$

For example, in this notation the one-loop dilatation generator [5] is given by

$$\mathbf{H}_0 = \mathbf{Q}_{2,0} = (\{\} - \{1\}) . \tag{2.7}$$

This notation is useful due to the nature of maximal scalar diagrams as discussed in [12]: an interaction of scalars at ℓ-loops with the maximal number of $2+2\ell$ legs can be composed of ℓ crossings of scalar lines. In the planar limit, the crossings correspond to the elementary permutations and at ℓ-loops there should be no more than ℓ permutations. In field theory this is a feature of maximal diagrams, but here we will assume the pattern to hold in general. Furthermore, a general feature of ordinary (one-loop) spin chains is that the r-th charge at leading order can be constructed from $r-1$ copies of the hamiltonian density which, in this case (2.7), is essentially an elementary permutation. We will therefore assume the contributions to the charges to be of the form

$$\mathbf{Q}_{r,2\ell-2} \sim \{p_1, \ldots, p_m\} \qquad \text{with } m \le r+\ell-2 \text{ and } 1 \le p_i \le r+\ell-2\,. \tag{2.8}$$

Finally, the even (odd) charges should be parity even (odd) and (anti)symmetric.[7] Parity $\mathfrak{p}$ acts on the interactions as

$$\mathfrak{p}\,\{p_1, \ldots, p_m\}\,\mathfrak{p}^{-1} = \{-p_1, \ldots, -p_m\}\,, \tag{2.9}$$

whereas symmetry acts as

$$\{p_1, \ldots, p_m\}^{\mathsf{T}} = \{p_m, \ldots, p_1\}\,. \tag{2.10}$$

Symmetry will ensure that the eigenvalues of the charges are real.

We can now write the dilatation operator up to three loops as given in [12]

$$\begin{aligned}
\mathbf{H}_0 &= \{\} - \{1\}\,,\\
\mathbf{H}_2 &= -2\{\} + 3\{1\} - \frac{1}{2}\left(\{1,2\} + \{2,1\}\right)\,,\\
\mathbf{H}_4 &= \frac{15}{2}\{\} - 13\{1\} + \frac{1}{2}\{1,3\} +\\
&\quad +3\left(\{1,2\} + \{2,1\}\right) - \frac{1}{2}\left(\{1,2,3\} + \{3,2,1\}\right)\,.
\end{aligned} \tag{2.11}$$

[6] This notation was introduced in [12] where one can find a set of rules for the simplification of involved expressions.

[7] In fact, the hamiltonian $\mathbf{H}(g)$ and charges $\mathbf{Q}_r(g)$ should be hermitian. The coefficients of the interaction structures should therefore be real (imaginary) for even (odd) r. Reality of the hamiltonian follows from the equivalence of the hamiltonian for the $\mathfrak{su}(2)$ sector and its conjugate.

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The one-loop contribution has been computed explicitly in field theory [5]. To obtain the higher-loop contributions it is useful to rely on the (quantitative) BMN limit, this suffices for the two-loop contribution [12]. At three-loops the BMN limit fixes all but a single coefficient. It can be uniquely fixed if, in addition, integrability is imposed [12].[8] The same is true at four-loops [17] and five-loops [53], the BMN limit and integrability *uniquely* fix the hamiltonian. We present the five-loop hamiltonian in table A.1 in appendix A.1. Expressions for the higher charges can be found in [53] along with a set of `Mathematica` routines to compute scaling dimensions explicitly and commutators of charges in an abstract way.

There are some interesting points to be mentioned regarding this solution. First of all, integrability and the thermodynamic limit fix exactly the right number of coefficients for a unique solution (up to five loops). Moreover we can give up on the quantitative BMN limit and only require proper scaling behavior. This merely allows for two additional degrees of freedom and the most general hamiltonian would be given by $\mathbf{H}'(g) = c_1\mathbf{H}(c_2 g)$. The constants c_1, c_2 correspond to symmetries of the defining equations, they can therefore not be fixed by algebraic arguments, but the quantitative BMN limit requires $c_1 = c_2 = 1$. For this solution, the contribution $\delta\mathbf{D}_n$ of one excitation of mode n to the scaling dimension in the BMN limit is given by

$$\delta\mathbf{D}_n = c_1\left(\sqrt{1 + c_2^2\,\lambda'\,n^2} - 1\right) + \mathcal{O}\left(g^{12}\right). \tag{2.12}$$

where the BMN coupling constant λ' is defined as

$$\lambda' = 8\pi^2\frac{g^2}{J^2} \qquad \text{i.e.} \qquad \lambda' = \frac{\lambda}{J^2}\,, \tag{2.13}$$

and J has been defined in (2.2). It is interesting to observe that the BMN square-root formula (2.44) for the energy of one excitation is predicted correctly. It would be important to better understand this intriguing interplay between integrability, and (qualitative and quantitative) thermodynamic scaling behavior. See also [52].

There is however one feature of the dilatation operator which cannot be accounted for properly. For increasing loop order ℓ the length of the interaction, $\ell+1$, grows. For a fixed length L of a state, the interaction is longer than the state when $\ell \geq L$. In this case the above hamiltonian does not apply, it needs to be extended by *wrapping interactions* which couple only to operators of a fixed length. In planar field theory these terms exist, they correspond to Feynman diagrams which fully encircle the state. We will comment on this kind of interactions in section 4.3. Here we only emphasize that $\mathbf{Q}_r$ is reliable only up to and including $\mathcal{O}\left(g^{2L-2r}\right)$.

2.2 Long-range Bethe ansatz

Minahan and Zarembo have demonstrated the equivalence of the one-loop, planar dilatation operator in the $\mathfrak{su}(2)$ subsector with the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain [5]. The discovery

[8] Alternatively one may use superconformal invariance [15] or the input from the conjectured result of [13], based on the rigorous computation of [14], all of which are compatible with (2.11).

of integrability opens up an alternative way to compute energies, namely by means of the algebraic Bethe ansatz. Serban and one of us have recently shown how to extend the Bethe ansatz to account for up to three-loop contributions [18]. This ansatz is based on the Inozemtsev spin chain [19, 20] after a redefinition of the coupling constant and the charges. For the Inozemtsev spin chain there exists an asymptotic[9] Bethe ansatz. It makes use of the Bethe equations

$$\exp(iLp_k) = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i}. \tag{2.14}$$

The left hand side is a free plane wave phase factor for the k-th magnon, with momentum p_k, going around the chain. The right hand side is "almost" one, except for a sequence of *pairwise, elastic* interactions with the $M-1$ other magnons, leading to a small phase shift. Without this phase shift, the equation simply leads to the standard momentum quantization condition for a free particle on a circle. The details of the exchange interactions are encoded into the functions $\varphi(p_k)$, and definitely change from model to model, but the two-body nature of the scattering is the universal feature leading to integrability. It allows the reduction of an M-body problem to a sequence of two-body problems. The energy and higher charge eigenvalues[10] are then given by the linear sum of contributions from the individual magnons

$$\mathbf{Q}_r = \sum_{k=1}^{M} \mathbf{q}_r(p_k)\,, \qquad \mathbf{H} = \mathbf{Q}_2\,. \tag{2.15}$$

Again, this additive feature is due to the nearly complete independence of the individual excitations. However, the details of the contribution of an individual excitation to the r-th charge, $\mathbf{q}_r(p_k)$, depend once more on the precise integrable model. For example, the $\mathrm{XXX}_{1/2}$ Bethe ansatz is obtained by setting

$$\varphi(p) = \frac{1}{2}\cot\left(\frac{1}{2}p\right), \qquad \mathbf{q}_r(p) = \frac{2^r}{r-1}\sin\left(\frac{1}{2}(r-1)p\right)\sin^{r-1}\left(\frac{1}{2}p\right). \tag{2.16}$$

The Bethe roots or *rapidities* φ_k, defined as $\varphi_k = \varphi(p_k)$ are also denoted by λ_k or u_k in the literature. The inversion of (2.16)

$$\exp(ip) = \frac{\varphi + \frac{i}{2}}{\varphi - \frac{i}{2}}, \qquad \mathbf{q}_r(\varphi) = \frac{i}{r-1}\left(\frac{1}{(\varphi + \frac{i}{2})^{r-1}} - \frac{1}{(\varphi - \frac{i}{2})^{r-1}}\right) \tag{2.17}$$

leads to the common and fully algebraic description in terms of rapidities φ_k instead of momenta p_k.

[9] *Asymptotic* refers to the fact that wrapping interactions are probably not taken into account correctly, see also section 4.3.

[10] Note that here and in the rest of the paper we will, somewhat loosely, avoid to notionally or, at times, semantically distinguish between operators and their eigenvalues, as it should always be clear from the context what is meant. If it is not, the statement should be true in both interpretations.

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The phase relation $\varphi(p) = \varphi(p, g)$ of the modified Inozemtsev spin chain is given by [18]

$$\begin{aligned}\varphi(p) &= \frac{1}{2}\cot\left(\frac{1}{2}p\right)\left(1 + 4g^2\sin^2\left(\frac{1}{2}p\right) - 8g^4\sin^4\left(\frac{1}{2}p\right) + \right.\\ &\qquad\left. + 8g^6\sin^4\left(\frac{1}{2}p\right) + 16g^6\sin^6\left(\frac{1}{2}p\right) + \cdots\right)\end{aligned} \tag{2.18}$$

and the single-excitation energy is

$$\mathbf{q}_2(p) = 4\sin^2\left(\frac{1}{2}p\right) - 8g^2\sin^4\left(\frac{1}{2}p\right) + 32g^4\sin^6\left(\frac{1}{2}p\right) - 160g^6\sin^8\left(\frac{1}{2}p\right) + \cdots. \tag{2.19}$$

This reproduces scaling dimensions in gauge theory up to three loops, $\mathcal{O}(g^6)$, when the dilatation operator $\mathbf{D}$ is identified with the spin chain hamiltonian $\mathbf{H}$ as follows

$$\mathbf{D}(g) = L + g^2\mathbf{H}(g), \qquad \text{with } \mathbf{H} = \mathbf{Q}_2 = \sum_{k=1}^{M} \mathbf{q}_2(p_k)\,. \tag{2.20}$$

Starting at four loops[11] it was noticed that the scaling dimensions do not obey BMN scaling. As this was an essential input for the construction of the model in section 2.1, the energies of the modified Inozemtsev spin chain cannot agree with the model. The breakdown of BMN scaling can be traced back to the term proportional to $g^6\sin^4(\frac{1}{2}p)$ in (2.18). While all the other terms are of $\mathcal{O}(J^0)$ in the BMN limit, this one is of order J^2.

Our aim is to find a Bethe ansatz for the spin chain model described in section 2.1, therefore we shall make an ansatz for $\varphi(p), \mathbf{q}_2(p)$ which is similar to (2.18), but which manifestly obeys BMN scaling

$$\begin{aligned}\varphi(p) &= \frac{1}{2}\cot\left(\frac{1}{2}p\right)\sum_{\ell=1}^{\infty}\alpha_\ell\,\sin^{2\ell-2}\left(\frac{1}{2}p\right)\,g^{2\ell-2}\,,\\ \mathbf{q}_2(p) &= \sum_{\ell=1}^{\infty}\beta_\ell\,\sin^{2\ell}\left(\frac{1}{2}p\right)\,g^{2\ell-2}\,.\end{aligned} \tag{2.21}$$

By comparison to (2.18), (2.19) we find $\alpha_{1,2,3} = +1, +4, -8$ and $\beta_{1,2,3} = +4, -8, +32$. Interestingly, a comparison of energies at four and five loops with our spin chain model shows that we can indeed achieve agreement (see section 2.5 and appendix A). The correct coefficients are $\alpha_{4,5} = +32, -160$ and $\beta_{4,5} = -160, +896$. Now it is not hard to guess, by "physicist's induction", analytic expressions for the phase relation

$$\varphi(p) = \frac{1}{2}\cot\left(\frac{1}{2}p\right)\sqrt{1 + 8g^2\sin^2\left(\frac{1}{2}p\right)} \tag{2.22}$$

and the magnon energy

$$\mathbf{q}_2(p) = \frac{1}{g^2}\left(\sqrt{1 + 8g^2\sin^2\left(\frac{1}{2}p\right)} - 1\right) \tag{2.23}$$

[11] Order $\mathcal{O}(g^8)$ in $\mathbf{D}(g) = L + g^2\mathbf{H}(g)$ correspond to $\mathcal{O}(g^6)$ in $\mathbf{H}(g)$.

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which agree up to $\ell = 5$ in (2.21). Furthermore, we have also found a generalization of the higher charges (2.16) to higher loops

$$\mathbf{Q}_r = \sum_{k=1}^{M} \mathbf{q}_r(p_k)\,, \qquad \mathbf{q}_r(p) = \frac{2\sin\left(\frac{1}{2}(r-1)p\right)}{r-1}\left(\frac{\sqrt{1+8g^2\sin^2\left(\frac{1}{2}p\right)}-1}{2g^2\sin\left(\frac{1}{2}p\right)}\right)^{r-1}. \tag{2.24}$$

A non-trivial consistency check of the conjectured all-order expressions (2.22), (2.23), (2.24) will be performed in section 3 where we will find a remarkable link to the predictions of semi-classical string theory. Note that the one-particle momentum does not depend on the coupling g and is given by $\mathbf{q}_1(p, g) = p$. The charges can be summed up into the "local" part of the transfer matrix

$$\mathbf{T}(x) = \exp i \sum_{r=1}^{\infty} x^{r-1}\,\mathbf{Q}_r + \cdots\,, \tag{2.25}$$

and we find its eigenvalue from (2.24) to be

$$\mathbf{T}(x) = \prod_{k=1}^{M} \frac{x - \frac{2g^2\exp(+\frac{i}{2}p_k)\sin(\frac{1}{2}p_k)}{\sqrt{1+8g^2\sin^2(\frac{1}{2}p_k)}-1}}{x - \frac{2g^2\exp(-\frac{i}{2}p_k)\sin(\frac{1}{2}p_k)}{\sqrt{1+8g^2\sin^2(\frac{1}{2}p_k)}-1}} + \cdots\,. \tag{2.26}$$

The transfer matrix at $x = 0$ gives the total phase shift along the chain

$$\mathbf{U} = \mathbf{T}(0) = \prod_{k=1}^{M} \exp(ip_k)\,, \tag{2.27}$$

which should equal $\mathbf{U} = 1$ for gauge theory states with cyclic symmetry. The dots in (2.25), (2.26) indicate further possible terms like x^L or g^{2L} which cannot be seen for the lower charges or at lower loop orders. It is possible that finding these terms will allow for a modification of the asymptotic Bethe equations (2.14) so as to make them exact, i.e. they would then correctly take into account the gauge theoretic wrapping interactions. In section 4 we will suggest that the latter are responsible for the recently observed gauge-string disagreements for long operators [33, 34, 18].

Our long-range Bethe equations (2.14), (2.22), along with the expressions (2.24) for the charge densities, look somewhat involved. We will now show in section 2.3 that they may be significantly simplified by fully eliminating the momentum variables p_k and expressing them through rapidity variables φ_k. In particular, this replaces all trigonometric expressions by rational or algebraic ones. Furthermore, it uncovers the remarkable analytic structure of the ansatz, which, interestingly, largely survives in the thermodynamic limit, cf. section 3. What is more, it allows us to find an intriguing link to *inhomogeneous* spin chains, as we shall elaborate in section 2.4. Apart from its potential conceptual importance, the last observation allows one to also correctly treat certain "singular" solutions of the Bethe ansatz, as will be explained in some detail in appendix A.3. After these conceptual elaborations we will proceed in section 2.5 to actually *test* that our long-range Bethe ansatz indeed properly diagonalizes the hamiltonian proposed in section 2.1. An application to the near-BMN limit is presented in section 2.6.

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2.3 The rapidity plane

The Bethe equation (2.14) involves momenta p_k on the left hand side and rapidities φ_k on the right hand side. The relation $\varphi(p)$ defined in (2.22) specifies the precise nature of the model. In the previous section we have used the momenta p_k as the fundamental variables and $\varphi_k = \varphi(p_k)$ as derived variables. Here we would like to take the opposite point of view and consider φ_k as fundamental. For that purpose we need to invert the relation (2.22), there turns out to be a remarkably simple form

$$\exp(ip) = \frac{x(\varphi + \frac{i}{2})}{x(\varphi - \frac{i}{2})} \tag{2.28}$$

where[12]

$$x(\varphi) = \frac{1}{2}\varphi + \frac{1}{2}\sqrt{\varphi^2 - 2g^2}\,. \tag{2.29}$$

The Bethe equations (2.14) can now be conveniently written without trigonometric functions as

$$\frac{x(\varphi_k + \frac{i}{2})^L}{x(\varphi_k - \frac{i}{2})^L} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{\varphi_k - \varphi_j + i}{\varphi_k - \varphi_j - i} \tag{2.30}$$

in great similarity with the Bethe ansatz for the Heisenberg model. In the new variables $\mathbf{T}(x)$ simplifies drastically to

$$\mathbf{T}(x) = \prod_{k=1}^{M} \frac{x - x(\varphi_k + \frac{i}{2})}{x - x(\varphi_k - \frac{i}{2})} + \cdots\,. \tag{2.31}$$

The local charges $\mathbf{Q}_r$ follow from the expansion (2.25)

$$\mathbf{Q}_r = \sum_{k=1}^{M} \mathbf{q}_r(\varphi_k)\,, \qquad \mathbf{q}_r(\varphi) = \frac{i}{r-1}\left(\frac{1}{x(\varphi + \frac{i}{2})^{r-1}} - \frac{1}{x(\varphi - \frac{i}{2})^{r-1}}\right). \tag{2.32}$$

It is interesting to see that the transfer matrix $\mathbf{T}(x)$ is not the obvious guess related to the Bethe equation (2.30). In analogy to the Heisenberg model, see e.g. [54], the immediate guess $\bar{\mathbf{T}}(\varphi)$ would be

$$\bar{\mathbf{T}}(\varphi) = \left(x\left(\varphi + \frac{i}{2}\right)\right)^L \prod_{k=1}^{M} \frac{\varphi - \varphi_k - i}{\varphi - \varphi_k} + \left(x\left(\varphi - \frac{i}{2}\right)\right)^L \prod_{k=1}^{M} \frac{\varphi - \varphi_k + i}{\varphi - \varphi_k}\,. \tag{2.33}$$

The Bethe equations (2.30) follow from the cancellation of poles at $\varphi = \varphi_j$. The charges $\bar{\mathbf{Q}}_r$ obtained from $\bar{\mathbf{T}}(\varphi)$ as $\bar{\mathbf{T}}(\varphi + \frac{i}{2})/x(\varphi + i)^L = \exp i \sum_{r=1}^{\infty} \varphi^{r-1} \bar{\mathbf{Q}}_r + \cdots$

$$\bar{\mathbf{Q}}_r = \sum_{k=1}^{M} \frac{i}{r-1}\left(\frac{1}{(\varphi_k + \frac{i}{2})^{r-1}} - \frac{1}{(\varphi_k - \frac{i}{2})^{r-1}}\right). \tag{2.34}$$

[12]Note that $x(\varphi)$ is odd under $\varphi \mapsto -\varphi$. This property is more manifest if we replace the square root by $\varphi\sqrt{1 - 2g^2/\varphi^2}$ which also straightforwardly yields the correct perturbative expansion for small g.

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In perturbation theory we can relate these charges to the physical charges $\mathbf{Q}_r$ by

$$\mathbf{Q}_r = \bar{\mathbf{Q}}_r + \frac{1}{2}(r+1)g^2\bar{\mathbf{Q}}_{r+2} + \frac{1}{8}(r+2)(r+3)g^4\bar{\mathbf{Q}}_{r+4} + \cdots \tag{2.35}$$

The form of these charges on an operatorial level is not clear to us. At one-loop they agree with the physical charges, but at higher loops their range grows twice as fast with the loop order as for $\mathbf{Q}_r$. Therefore $\bar{\mathbf{Q}}_2$ is clearly not a suitable candidate for the dilatation operator $\mathbf{D}$. In the next section we will find a natural explanation using, however, a different basis.

Let us comment on (2.29) and its inverse

$$\varphi(x) = x + \frac{g^2}{2x}\,. \tag{2.36}$$

The map between x and φ is a double covering map. For every value of φ there are two corresponding values of x, namely

$$\varphi \longleftrightarrow \left\{x, \frac{g^2}{2x}\right\}. \tag{2.37}$$

For small values of g, where the Bethe ansatz describes the long-range spin chain, we will always assume that $x \approx \varphi$. When g is taken to be large (if this makes sense at all is a different question), however, special care is needed in selecting the appropriate branch. The double covering map for x and φ has an analog for the transfer matrices $\mathbf{T}(x)$ and $\bar{\mathbf{T}}(\varphi)$. We find the relation

$$\frac{\mathbf{T}(x)\,\mathbf{T}(g^2/2x)}{\mathbf{T}(0)} \approx \frac{\bar{\mathbf{T}}(\varphi(x)+\frac{i}{2})}{x(\varphi(x)+i)^L}\,. \tag{2.38}$$

which holds if the second term in (2.33) is dropped. It can be proved by using the double covering relation

$$(x-x')\left(1-\frac{g^2}{2xx'}\right) = \left(x+\frac{g^2}{2x}\right) - \left(x'+\frac{g^2}{2x'}\right) = \varphi - \varphi'\,. \tag{2.39}$$

We believe it is important to further study the implications of the double covering maps. This might lead to insight into the definition of our model, possibly even beyond wrapping order.

2.4 The inhomogeneous Bethe ansatz

The equations (2.30), (2.33) are very similar to the Bethe ansatz for an inhomogeneous spin chain, see e.g. [54].[13] The only difference is that the inhomogeneous Bethe ansatz requires a polynomial of degree L in φ whereas the function $x(\varphi)^L$ also contains negative powers of φ (when expanded for small g). In fact, for the function $x(\varphi)$ as defined in (2.29), the negative powers only start at g^{2L} and are irrelevant for the desired accuracy of our asymptotic model. In order to relate our equations to a well-known model we could truncate the

[13] See appendix B for a reformulation of more general long-range spin chains, in particular the Inozemtsev spin chain, in terms of an inhomogeneous spin chain.

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expansion of $x(\varphi)^L$ at $\mathcal{O}\left(g^{2L}\right)$. Remarkably this truncation can be achieved analytically: the inverse (2.36) can be used to show that

$$P_L(\varphi) = x(\varphi)^L + \left(\frac{g^2}{2x(\varphi)}\right)^L \tag{2.40}$$

is a polynomial of degree L in φ, which is exactly the proposed truncation at $\mathcal{O}\left(g^{2L}\right)$. In fact, the polynomial is easily factorized as follows

$$P_L(\varphi) = \prod_{p=1}^{L}(\varphi - \phi_p) \quad \text{with} \quad \phi_p = \sqrt{2}\, g \cos\frac{\pi(2p-1)}{2L}\,. \tag{2.41}$$

We now replace $x(\varphi)^L$ by $P_L(\varphi)$ in the Bethe equation (2.30)

$$\frac{P_L(\varphi_k + \frac{i}{2})}{P_L(\varphi_k - \frac{i}{2})} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{\varphi_k - \varphi_j + i}{\varphi_k - \varphi_j - i} \tag{2.42}$$

and transfer matrix (2.33)

$$\bar{\mathbf{T}}(\varphi) = P_L\left(\varphi + \frac{i}{2}\right)\prod_{k=1}^{M}\frac{\varphi - \varphi_k - i}{\varphi - \varphi_k} + P_L\left(\varphi - \frac{i}{2}\right)\prod_{k=1}^{M}\frac{\varphi - \varphi_k + i}{\varphi - \varphi_k}\,, \tag{2.43}$$

and obtain the Bethe ansatz for an inhomogeneous spin chain with shifts ϕ_p.

Let us first of all comment on the inhomogeneities. Our spin chain is homogeneous, how can the Bethe ansatz of an inhomogeneous spin chain describe our model? The equation (2.35) relates a homogeneous charge $\mathbf{Q}_r$ on the left hand side with inhomogeneous charges $\bar{\mathbf{Q}}_s$ on the right hand side. A possible resolution may be found by investigating the inhomogeneous spin chain itself: on the one hand, the order of the inhomogeneities ϕ_p does not matter for the Bethe ansatz and thus for the eigenvalues of the charges. On the other hand, it should certainly influence the eigenstates. Consequently, the eigenstates should be related by a similarity transformation[14] and (2.35) is merely an equation for the eigenvalues of the involved operators. Alternatively, the $\bar{\mathbf{Q}}_s$ in (2.35) can be interpreted as homogenized charges which are related to the naive, inhomogeneous charges $\bar{\mathbf{Q}}_s$ by a change of basis. To understand our model better, it would be essential to investigate this point further and find the map between our homogeneous spin chain model and the common inhomogeneous spin chain.

Now we have totally self-consistent Bethe equations (2.42) but the physical transfer matrix $\mathbf{T}(x)$ as defined in (2.31) (consequently the energy $\mathbf{D}$ and charges $\mathbf{Q}_r$) does not agree with the inhomogeneous transfer matrix $\bar{\mathbf{T}}(\varphi)$. The physical transfer matrix involves the function $x(\varphi)$. When the coupling constant g is arbitrary, this function is ambiguous due to the two branches of the square root. This is not a problem in perturbation theory, however, even there inconsistencies are observed at higher order in g, see appendix A.3. Remarkably,

[14] The inhomogeneities ϕ_p and ϕ_{p+1} can be interchanged by conjugation with $R_{p,p+1}(\phi_p - \phi_{p+1})$, we thank K. Zarembo for a discussion on this point.

these appear precisely at the order where wrapping interactions start to contribute and our asymptotic Bethe ansatz is fully consistent to the desired accuracy. Conversely, there are signs of the missing of wrapping terms. We hope that finding a cure for the problems beyond wrapping order might help to find a generalization of the Bethe equations which include wrapping interactions. Presumably these equations will have a substantially different form.

2.5 Comparison

It is obviously important to check that our asymptotic Bethe ansatz as developed in sections 2.2, 2.3 and 2.4 indeed properly diagonalizes the original, rather involved five-loop hamiltonian (2.11), table A.1 which is currently known up to five loops. This is particularly important as asymptotic Bethe ansätze such as ours are usually not easy to derive rigorously. However, we can certainly check its exactness by comparing its predictions to the results obtained by brute-force diagonalization of the hamiltonian for specific states. Here we summarize the results of the comparisons we have performed, and refer the reader to appendix A and table 1 for further details.

Two excitations. The perturbative Bethe ansatz gives results for two-excitation states of arbitrary length away from the near BMN limit (see appendix A.2 for details). The structure of the energy agrees with the conjectured formula (A.5), see also [17], and the coefficients agree at five-loop accuracy.

Three excitations. For three excitations, there exist paired and unpaired solutions. The unpaired three-excitation states are singular and a direct computation requires the Bethe ansatz of section 2.4. Alternatively, their energies can be computed via *mirror solutions.* The fact that both methods (see appendix A.3) lead to the same result hints at the consistency of our equations. The paired solutions for $L = 7$ and $L = 8$ agree with the perturbative gauge theory results (cf. appendix A.4 for details). Finding the Bethe roots for longer spin chains becomes more and more involved.

More excitations. All states with up to length $L \leq 8$ and a few examples up to length 10 have been computed in the Bethe ansatz. Their energies and charges agree with the eigenvalues of the spin chain operators.

Higher charges. For the afore mentioned states we have also computed the first few orders of the higher charges $\mathbf{Q}_{3,4,5,6}$, using our Bethe ansatz, and compared them to the direct diagonalization of the long-range spin chain model. We find agreement for all instances of states with low excitation numbers.

BMN limit. The general BMN energy formula

$$\mathbf{D}(\lambda') - J = \sum_{k=1}^{M} \sqrt{1 + \lambda' n_k^2} \tag{2.44}$$

is easily confirmed. Regarding the single excitation energy formula (2.23) this is not a miracle. However, it is fascinating to have found an integrable model where the proper

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BMN behavior is more or less *implemented.* This indicates that there might be a deeper connection between integrability and the BMN/planar limit.

Conclusion. In conclusion, we can say that for all considered examples (including all states of length L up to 8) the Bethe ansatz yields precisely the same spectrum as the hamiltonian approach described in section 2.1. It shows that an integrable spin chain of infinitely long-range, and with a well-defined thermodynamic limit, is very likely to exist, largely putting to rest the concerns expressed in [18]. These were based on arguments [20] that the elliptic Inozemtsev chain should be the most general integrable model, paralleling analogous results for the Calogero-Moser multi-particle system. However, the proof seems to implicitly assume that the lowest charge only contains two-spin interactions, whereas our new chain definitely is not of this type (see again [18]). In terms of the Bethe ansatz there may seem to be many such models. These would be obtained by appropriately modifying the coefficients in (2.21). If we however demand that the model is related to an inhomogeneous spin chain as in section 2.4 we find a unique model with thermodynamic scaling behavior, see appendix B.

The upshot for the integrable spin chain model is similar: in its construction we have assumed a very specific form of interactions and the obtained hamiltonian has turned out to be unique (at five loops). In other words, the very relations (2.22), (2.23) are special and correspond to the assumed form of interactions (ii).[15] At any rate, these relations are very suggestive in view of a correspondence to string theory on plane waves. It is therefore not inconceivable that our Bethe ansatz does indeed asymptotically describe planar $\mathcal{N}=4$ gauge theory in the $\mathfrak{su}(2)$ subsector at higher-loops.

2.6 The near-BMN limit

Let us now use our novel ansatz to obtain all-orders predictions for the $1/J$ corrections to the anomalous dimensions of BMN type operators, i.e. let us consider the so-called near-BMN limit. In fact, in the first non-trivial case of states with two excitations an all-loop gauge theory expression for this correction has been guessed in [17]. Excitingly, we shall find that our ansatz precisely reproduces this conjecture! The expression in question, which agrees with (A.5) and with table A.6 at five-loops, is

$$\mathbf{D}(J,n,\lambda')=J+2\sqrt{1+\lambda'\,n^2}-\frac{4\lambda'\,n^2}{J\sqrt{1+\lambda'\,n^2}}+\frac{2\lambda'\,n^2}{J(1+\lambda'\,n^2)}+\mathcal{O}\left(\frac{1}{J^2}\right) \tag{2.45}$$

where J and λ' have been defined in, respectively, (2.2) and (2.13). The first $1/J$ term can be regarded as a renormalization of the term $\lambda' n^2$ in the first square root. For instance, we might replace J in the definition of λ' by $J+4$ to absorb the second term into the leading order energy. Unfortunately, as has already been pointed out in [17], this formula does not agree with the expression for the near plane-wave limit on the string side derived in [33, 34]

$$\mathbf{D}(J,n,\lambda')=J+2\sqrt{1+\lambda'\,n^2}-\frac{2\lambda'\,n^2}{J}+\mathcal{O}\left(\frac{1}{J^2}\right). \tag{2.46}$$

[15] This picture is rather similar to the Inozemtsev spin chain where the requirement of pairwise interactions of spins at a distance was shown to lead to the phase relation of the Inozemtsev Bethe ansatz.

Now let us compute the momenta p_k for the case $M = 2$ and in the near BMN limit. That is the momenta are expanded around $J = \infty$:

$$p = \frac{p^{(0)}}{J} + \frac{p^{(2)}}{J^2} + \frac{p^{(4)}}{J^3} + \cdots . \tag{2.47}$$

Then the Bethe equations (2.14) are solved order by order in $1/J$. In general, we would have to start with two distinct Bethe equations, which determine the two roots φ_1, φ_2. However, the momentum constraint $\mathbf{U} = 1$ (2.27) implies in this case a symmetric distribution of the roots in the complex plane [5]: $\varphi \equiv \varphi_1 = -\varphi_2$, i.e. $p \equiv p_1 = -p_2$. We are thus left with one root determined by

$$\exp\left(ip(J+2)\right) = \frac{\cot\left(\frac{1}{2}p\right)\sqrt{1 + \lambda' J^2 \pi^{-2} \sin^2\left(\frac{1}{2}p\right)} + i}{\cot\left(\frac{1}{2}p\right)\sqrt{1 + \lambda' J^2 \pi^{-2} \sin^2\left(\frac{1}{2}p\right)} - i} . \tag{2.48}$$

The general solution of this equation at leading order in $1/J$ is then given by $p^{(0)} = 2\pi n$ where n is an integer. Substituting this back into (2.47) and expanding up to $\mathcal{O}\left(1/J^2\right)$ yields the first correction to the momentum

$$p^{(2)} = -\frac{2n\pi(2\sqrt{1+\lambda' n^2} - 1)}{\sqrt{1+\lambda' n^2}} . \tag{2.49}$$

The conjectured near-BMN energy formula (2.45) is then indeed obtained by inserting the perturbed momentum into (2.20) with (2.23) and expanding the result in $1/J$.

In view of recent results which might soon also allow the computation of the $1/J^2$ corrections on the string side [55], one easily obtains in much the same way the next order gauge correction to the momentum

$$p^{(4)} = \frac{2\pi n(-4 - 6\lambda' n^2 - 2\lambda'^2 n^4)}{(1+\lambda' n^2)^{5/2}} + \frac{2\pi n(5 + 8\lambda' n^2 + 4\lambda'^2 n^4)}{(1+\lambda' n^2)^2} \tag{2.50}$$

and the energy now reads

$$\begin{aligned} \mathbf{D}(J, n, \lambda') \;=\; & J + 2\sqrt{1 + \lambda'\, n^2} - \frac{4\lambda'\, n^2}{J\sqrt{1+\lambda'\, n^2}} + \frac{2\lambda'\, n^2}{J(1+\lambda'\, n^2)} \\ & + \frac{15\lambda' n^2 + 20\lambda'^2 n^4 + 8\lambda'^3 n^6}{J^2(1+\lambda' n^2)^{5/2}} - \frac{n^2\pi^2(\lambda' n^2 + 2\lambda'^2 n^4 + \lambda'^3 n^6)}{3J^2(1+\lambda' n^2)^{5/2}} \\ & - \frac{12\lambda' n^2 + 16\lambda'^2 n^4 + 4\lambda'^3 n^6}{J^2(1+\lambda' n^2)^3} + \mathcal{O}\left(\frac{1}{J^3}\right) . \end{aligned} \tag{2.51}$$

Of course, agreement with string theory is not expected beyond, at most, two loops.

3. Stringing spins and spinning strings at all loops

3.1 Perturbative gauge theory: the thermodynamic limit

The thermodynamic limit is the limit in which the length of the spin chain L as well as the number of excitations M is taken to infinity while focusing on the the low-energy spectrum. In this limit, it was observed that the r-th charge $\mathbf{q}_{r,0}$ of one magnon (2.24) at one-loop scales as L^{-r} [29]. Here, we would like to generalize the thermodynamic limit to higher-loops. From the investigation of the closely related BMN limit as well as from classical spinning strings, we infer that each power of the coupling constant g should be accompanied by one power of $1/L$. We thus replace g according to

$$g^2 = \frac{\lambda}{8\pi^2} \mapsto L^2 g^2 \,, \tag{3.1}$$

where we have used the same symbol g for the rescaled, effective coupling constant in thermodynamic limit. It is common belief that this scaling behavior holds for perturbative gauge theory, but it is clearly not a firm fact. We shall assume its validity for several reasons: Firstly, it was not only confirmed at one-loop, but also at two-loops [56, 12]. It is a nice structure and conceptually it would be somewhat disappointing if broken at some higher loop order. Secondly, the AdS/CFT correspondence seems to suggest it. Thirdly, it will allow us to define charges uniquely, see [15, 29].

In conclusion, we expect that the scaling of charges in the thermodynamic limit is given by

$$\mathbf{q}_r(g) \mapsto L^{-r}\mathbf{q}_r(g) \,, \tag{3.2}$$

Due to the large number of excitations $M = \mathcal{O}(L)$, the total charge scales as $\mathbf{Q}_r(g) \mapsto L^{-r+1}\mathbf{Q}_r(g)$.[16] In particular the scaling dimension is

$$\mathbf{D}(g) \mapsto L\,\mathbf{D}(g) \qquad \text{with} \qquad \mathbf{D}(g) = 1 + g^2\,\mathbf{Q}_2(g) \,. \tag{3.3}$$

The relevant quantities of the Bethe ansatz should behave as follows:

$$\varphi_k \mapsto L\varphi_k \,, \qquad x(\varphi_k) \mapsto Lx(\varphi_k), \qquad p(\varphi_k) \mapsto p(\varphi_k)/L \,. \tag{3.4}$$

where[17]

$$x(\varphi) = \frac{1}{2}\varphi + \frac{1}{2}\sqrt{\varphi^2 - 2g^2} \qquad \text{and} \qquad p(\varphi) = \frac{1}{\sqrt{\varphi^2 - 2g^2}} \,. \tag{3.5}$$

In the scaling limit the charges (2.32) become

$$\mathbf{q}_r(\varphi) = \frac{1}{\sqrt{\varphi^2 - 2g^2}} \frac{1}{\left(\frac{1}{2}\varphi + \frac{1}{2}\sqrt{\varphi^2 - 2g^2}\right)^{r-1}} = \frac{p(\varphi)}{x(\varphi)^{r-1}} \,, \tag{3.6}$$

where the momentum and energy densities are, respectively,

$$\mathbf{q}_1(\varphi) = p(\varphi) = \frac{1}{\sqrt{\varphi^2 - 2g^2}} \qquad \text{and} \qquad \mathbf{q}_2(\varphi) = \frac{1}{\sqrt{\varphi^2 - 2g^2}} \frac{1}{\frac{1}{2}\varphi + \frac{1}{2}\sqrt{\varphi^2 - 2g^2}} \,. \tag{3.7}$$

[16] In the BMN limit the total charges scale as in (3.2). We recall that the difference between the BMN and the thermodynamic limit is that in the former the magnon number M stays finite.

[17] The correct sign for the square root $\sqrt{\varphi^2 - 2g^2}$ in perturbation theory is the same as of φ. More accurately we should write $\varphi\sqrt{1 - 2g^2/\varphi^2}$.

The discrete sums (2.15) over excitation charges turn into integrals over the distribution density

$$\rho(\varphi) = \frac{1}{L}\sum_{k=1}^{M}\delta(\varphi-\varphi_k) \qquad \text{or} \qquad \frac{1}{L}\sum_{k=1}^{M} f(\varphi_k) \mapsto \int_{\mathbf{C}} d\varphi\,\rho(\varphi)\,f(\varphi)\,, \tag{3.8}$$

with support on a discrete union of K smooth contours $\mathbf{C} = \mathbf{C}_1 \cup \cdots \cup \mathbf{C}_K$, i.e. the charges are given by

$$\mathbf{Q}_r = \int_{\mathbf{C}} d\varphi\,\rho(\varphi)\,\mathbf{q}_r(\varphi)\,. \tag{3.9}$$

Note the normalization of the density:

$$\int_{\mathbf{C}} d\varphi\,\rho(\varphi) = \alpha \qquad \text{with} \qquad \alpha = \frac{M}{L}\,, \tag{3.10}$$

where α is termed filling fraction.

The distribution density, in addition to the normalization condition (3.10), is subject to the momentum constraint

$$\mathbf{Q}_1 = \int_{\mathbf{C}} d\varphi\,\rho(\varphi)\,p(\varphi) = 2\pi m\,, \tag{3.11}$$

where m is an integer mode number as required by cyclic symmetry. Finally, the continuum Bethe equations derived from (2.14) lead to a system of singular integral equations, determining the distribution density $\rho(\varphi)$,

$$2\,\fint_{\mathbf{C}} \frac{d\varphi'\,\rho(\varphi')}{\varphi-\varphi'} = \frac{1}{\sqrt{\varphi^2-2g^2}} + 2\pi n_\nu \qquad \text{with} \qquad \varphi \in \mathbf{C}_\nu\,, \tag{3.12}$$

which are to hold on all K cuts $\mathbf{C}_1,\ldots,\mathbf{C}_K$, i.e. $\nu = 1,\ldots,K$. These equations are solved explicitly in appendix C.1.2 for the folded string and in appendix C.2.2 for the circular string. The charges are then determined from the solution $\rho(\varphi)$ by computing the integrals (3.9), using (3.6). The parameters n_ν are integer mode numbers obtained from taking the logarithm of (2.14). In fact, the right hand side of (3.12) is just $p(\varphi) + 2\pi n_\nu$, while the left hand side describes the factorized scattering of the excitations in the thermodynamic limit.[18] One easily checks that the perturbative expansion of the Bethe equations (3.12) and of the expressions for the energy (3.7) reproduces, by construction, the three-loop thermodynamic Inozemtsev-Bethe ansatz in [18].

Note that the Bethe equation (3.12) can alternatively be obtained as a consistency condition on the transfer matrix $\bar{\mathbf{T}}(u)$. In the thermodynamic limit, the transfer matrix becomes

$$\frac{\bar{\mathbf{T}}(\varphi)}{x(\varphi)^L}, \frac{\bar{\mathbf{T}}(\varphi)}{P_L(\varphi)} \to 2\cos\bar{\mathbf{G}}_{\text{sing}}(\varphi) \qquad \text{with} \qquad \bar{\mathbf{G}}_{\text{sing}}(\varphi) = \frac{1}{2\sqrt{\varphi^2-2g^2}} + \bar{\mathbf{G}}(\varphi) \tag{3.13}$$

[18]For more information on the mode numbers n_ν and m, as well as on certain subtleties involving root "condensates", we refer to the detailed one and two-loop discussion in [35], which largely generalizes to our all-loop equations. However, note in (3.12) the appearance of an additional square root cut in the potential $p(\varphi)$.

where $\bar{\mathbf{G}}(\varphi)$ is the singularity-free φ resolvent. The resolvent has many sheets, but the transfer matrix $2\cos\bar{\mathbf{G}}_{\text{sing}}(\varphi)$ must be single-valued on the complex φ plane. This requires

$$\bar{\mathbf{G}}_{\text{sing}}(\varphi+i\epsilon)+\bar{\mathbf{G}}_{\text{sing}}(\varphi-i\epsilon)=2\pi n \tag{3.14}$$

across a branch cut of $\bar{\mathbf{G}}$ at φ, which is an equivalent to the Bethe equation (3.12). At this point, it is however not clear how the physical transfer matrix $\mathbf{T}(x)$ is related to the physical resolvent $\mathbf{G}(x)$ and if there is also a consistency requirement which leads to the Bethe equations. This is largely related to mirror cuts in $\mathbf{T}(g^2/2x)$ which are due to the double covering map $x(\varphi)$.

3.2 Semi-classical string theory: the Bethe equation

In [35] Kazakov, Marshakov, Minahan and Zarembo developed a general approach for finding the semi-classical solutions of the string sigma model in the large charge limit. They astutely exploited the (classical) integrability of the sigma model and derived the equations determining the monodromy matrix of the system. These singular integral equations were termed "classical" Bethe equations in [35], and it was shown that the monodromy matrix may be interpreted as their resolvent. Not surprisingly, the derivation conceptionally differs from the one of the algebraic Bethe ansatz for quantum spin chains. In particular, one may introduce a pseudodensity $\sigma(x)$ as the imaginary part of the resolvent $\mathcal{G}(x)$ along the discontinuities $\mathcal{C}=\mathcal{C}_1\cup\ldots\cup\mathcal{C}_K$ in the complex plane of the spectral parameter x:

$$\mathcal{G}(x)=\int_{\mathcal{C}} dx'\,\frac{\sigma(x')}{x'-x}\,, \tag{3.15}$$

i.e. $\mathcal{G}(x)$ is manifestly analytic on the physical sheet of the complex x-plane, except for the set of cuts $\mathcal{C}$. The normalization condition of $\sigma(x)$ was found to be

$$\int_{\mathcal{C}} dx\,\sigma(x)\left(1-\frac{g^2}{2x^2}\right)=\alpha\,, \tag{3.16}$$

where the filling fraction $\alpha=M/L$ is defined in the same fashion as in section 3.1. Written in this form, we see that the function $\sigma(x)$ should not be interpreted as a distribution density of the local excitations in the spectral x plane (see, however, the next section 3.3). The local charges are then neatly expressed as

$$\mathcal{Q}_r=\int_{\mathcal{C}} dx\,\frac{\sigma(x)}{x^r}\,. \tag{3.17}$$

This means that the Taylor expansion of the string resolvent around $x=0$ generates these charges:

$$\mathcal{G}(x)=\sum_{r=1}^{\infty}\mathcal{Q}_r\,x^{r-1}\,. \tag{3.18}$$

As in gauge theory the first charge is the momentum $\mathcal{Q}_1$. It is subject to the integer constraint

$$\mathcal{Q}_1=2\pi m\,, \tag{3.19}$$

while the rescaled string energy is also precisely given by

$$\mathcal{E} = 1 + g^2 \mathcal{Q}_2(g)\,. \tag{3.20}$$

Finally the Bethe equation, which is solved in appendix C.1.1 and appendix C.2.1 for the folded and the circular string, respectively, reads

$$2 \fint_{\mathcal{C}} dx' \, \frac{\sigma(x')}{x - x'} = \mathcal{E}\, \frac{x}{x^2 - \frac{1}{2}g^2} + 2\pi n_\nu \qquad \text{with} \qquad x \in \mathcal{C}_\nu\,. \tag{3.21}$$

This ends our brief summary of the classical Bethe equations of [35]. In appendix D we present an equivalent set of equations using the true density of excitations ρ instead of the pseudodensity σ.

3.3 Structural matching of gauge and string theory

Let us structurally compare the gauge and string ansätze. The normalization condition (3.16) of string theory appears to be incompatible with the one in gauge theory (3.10) This may be fixed by relating the string and gauge spectral measures through

$$d\varphi = \left(1 - \frac{g^2}{2x^2}\right) dx\,. \tag{3.22}$$

Upon integration we recover the relation (2.29), (2.36) between the two spectral parameters φ and x from the study of the discrete system

$$\varphi = x + \frac{g^2}{2x}\,, \qquad x = \frac{1}{2}\varphi + \frac{1}{2}\sqrt{\varphi^2 - 2g^2}\,. \tag{3.23}$$

Interestingly, this is the same change of variables employed in [35] in order to show the two-loop agreement between string and gauge theory.[19] Here we see that the relationship should actually hold to *all* loops if we are to compare the two structures. One now easily checks the elegant formula, cf. (3.6),

$$\mathbf{q}_r(\varphi)\, d\varphi = \frac{dx}{x^r}\,, \tag{3.24}$$

i.e. the scaled gauge charge densities $\mathbf{q}_r(\varphi)$ in (3.6) and the string charge densities x^{-r} *precisely* agree for all r! Equation (3.24) is one of the key results of this paper, as it demonstrates the structural equivalence of the elementary excitations in string and gauge theory. We will have more to say about this at the end of section 4. The all-loop agreement between the infinite set of gauge and string theory charges could be established if one could show that the gauge theory distribution density $\rho_{\mathrm{g}}(\varphi)$ and the function $\sigma_{\mathrm{s}}(x(\varphi))$ coincide:

$$\rho_{\mathrm{g}}(\varphi) \stackrel{?}{=} \sigma_{\mathrm{s}}(x)\,. \tag{3.25}$$

This is however not the case, as was observed previously [18]. Indeed, as a first sign of trouble we observe that only at one loop the expansion point $\varphi = 0$ gets mapped to $x = 0$. For $g \neq 0$ the map (3.23) is singular and actually represents the φ-plane as a double cover

[19] A very similar change of variables also relates gauge theory and the generating function of Bäcklund charges, see [30].

of the x-plane. Under this map the gauge Bethe equation (3.12) turns into[20]

$$2\fint_{\mathcal{C}} dx'\,\frac{\sigma_{\rm g}(x')}{x-x'} = \frac{1}{x}\,\frac{1}{1-\frac{g^2}{2x^2}} + \frac{g^2}{x}\int_{\mathcal{C}} dx'\,\frac{\sigma_{\rm g}(x')}{x'^2}\,\frac{1}{1-\frac{g^2}{2xx'}} + 2\pi n_\nu\,. \tag{3.26}$$

Here the gauge pseudodensity is $\sigma_{\rm g}(x) := \rho(\varphi)$, while the string Bethe equation (3.21) may be rewritten as

$$2\fint_{\mathcal{C}} dx'\,\frac{\sigma_{\rm s}(x')}{x-x'} = \frac{1}{x}\,\frac{1}{1-\frac{g^2}{2x^2}} + \frac{g^2}{x}\int_{\mathcal{C}} dx'\,\frac{\sigma_{\rm s}(x')}{x'^2}\,\frac{1}{1-\frac{g^2}{2x^2}} + 2\pi n_\nu\,, \tag{3.27}$$

where we used (3.20). The only distinction between (3.26), (3.27) is the slightly different integrand of the integral on the right hand side of each equation. In this form the agreement of all charges up to exactly two loops is manifest.

The reader may find it instructive to also contemplate the form of the string Bethe equation (3.21) or (3.27) on the spectral φ-plane:

$$\begin{aligned} 2\fint_{\mathbf{C}} \frac{d\varphi'\,\rho_{\rm s}(\varphi')}{\varphi-\varphi'} &= \frac{1}{\sqrt{\varphi^2-2g^2}} + 2\pi n_\nu + 2g^2 \int_{\mathbf{C}} \frac{d\varphi'\rho_{\rm s}(\varphi')}{\sqrt{\varphi^2-2g^2}\sqrt{\varphi'^2-2g^2}} \times \\ &\quad\times \frac{\varphi-\sqrt{\varphi^2-2g^2}-\varphi'+\sqrt{\varphi'^2-2g^2}}{\left(\varphi+\sqrt{\varphi^2-2g^2}\right)\left(\varphi'+\sqrt{\varphi'^2-2g^2}\right)-2g^2}\,, \end{aligned} \tag{3.28}$$

where $\rho_{\rm s}(\varphi) := \sigma_{\rm s}(x)$. Comparing to the perturbative gauge Bethe equation (3.12) we notice that the difference is generated at $\mathcal{O}\left(g^4\right)$ (corresponding to three loop order) by the last term on the right hand side of (3.28). In this form it becomes very easy (for details, see appendix E) to prove in generality, for even (i.e. the odd charges are zero) solutions, the "curious observation" of [18, 30] which involved a conjecture about the structure of the three-loop disagreement between perturbative gauge and semi-classical string theory.

4. Resolution of the puzzle: a proposal

It is by now rather well established that there is a disagreement between gauge theory and string theory at three-loop order, unless the three-loop dilatation operator proposed in [12] and confirmed in [15] is incorrect. However, note the recent strong support for its validity from the conjecture [13] based on the three-loop computation of [14]. The disagreement first showed up in the near BMN limit [33, 34], and subsequently in the similar but different case of Frolov-Tseytlin (FT) spinning strings [18]. Assuming that the AdS/CFT correspondence is indeed correct and exact, a possible reason for the mismatch was first pointed out in [18]. Indeed, gauge and string theory employ slightly different scaling procedures. Here we will elaborate and significantly refine this possible explanation. Our discussion should apply equally well to the near BMN and FT situations. To be specific, we will use the BMN notation λ' in order to explain our argument.[21]

[20] The second integral can be interpreted as the effect of mirror cuts $\mathcal{C}' = g^2/2\mathcal{C}$ due to the double covering map $x(\varphi)$.

[21] Note the following subtlety: for the near BMN limit it is convenient to define this parameter as in (2.2), (2.13). In the spinning string discussion we should instead define $\lambda' = 8\pi^2 g^2/L^2$. In the strict BMN limit the difference is of course irrelevant.

$$
\begin{tikzcd}
\text{exact result} & G(\lambda', L) = F(\lambda, L) \arrow[r, "L \to \infty"] \arrow[d, "\text{expand in } \lambda"'] & G(\lambda') \arrow[d, "\text{expand in } \lambda'"] & \text{classical string theory} \\
\text{perturbative gauge theory} & F_l(L) \arrow[r, "L \to \infty"'] & F_l \neq G_l & \text{mismatch}
\end{tikzcd}
$$

Figure 1: A possible explanation for both the near BMN and the FT spinning strings disagreement. F_ℓ excludes gauge theory wrapping effects, while G_ℓ is expected to include them.

4.1 Order of limits

The comparison takes place in the thermodynamic limit $L \to \infty$ and in an expansion around $\lambda' = 0$. However starting with an exact function $F(\lambda, L)$, we must decide which limit is taken first. It turns out that for classical string theory, the thermodynamic limit $L \to \infty$ is a basic assumption. The resulting energy may then be expanded in powers of λ'. In contrast, gauge theory takes the other path. The computations are based on perturbation theory around $\lambda = 0$. This expansion happens to coincide with the expansion in λ' and for the thermodynamic limit one may drop subleading terms in $1/L$. The claim has been that the order of limits does potentially matter [18]. This is best illustrated in the noncommutative diagram figure 1. Semi-classical string theory corresponds to the upper right corner of the diagram, i.e. it requires the large spin limit. Conversely, perturbative gauge theory is situated at the lower left corner, where the length L is finite, but only the first few orders in λ are known. (However, we recall that the number of known terms grows with L, if our spin chain ansatz is correct.)

The BMN and FT proposals are both based on the assumption that the diagram in figure 1 does commute. In other words one should be able to compare, order by order, the gauge theory loop expansion with the string theory expansion in λ'. That this might in fact not be true was first hinted at in [57], where also an example was given. Another, more closely related, instance where the different limiting procedures lead to different results can be found in [18]. For the hyperbolic Inozemtsev spin chain it was shown that the order of limits does matter. In the "gauge theory" order, this spin chain appears to have no proper thermodynamic limit. For the "string theory" order, i.e. when the thermodynamic limit is taken right from the start, it is meaningful! (However, the resulting asymptotic Inozemtsev-Bethe ansatz also fails to reproduce the three-loop string results, cf. (59) of [18]).

In order to make contact with string theory we propose (in agreement with [57]) to sum up the perturbation series in λ before taking the thermodynamic limit. With the all-loop spin chain at hand this may indeed be feasible. In contrast to the Inozemtsev chain, there appears to be no difference between the two orders of limits (essentially because the thermodynamic limit is well-behaved in perturbation theory). However one has to take into account wrapping interactions. These arise at higher loop orders ℓ when the interaction stretches all around the state, i.e. when $\ell \geq L$. We will discuss them after the following example, which illustrates the potential importance of these interactions.

4.2 Example

Here we present an example where one can see the importance of the order of limits. We choose a function

$$F(\lambda, L) = \frac{\lambda^L}{(c+\lambda)^L} = \Big(1 + \frac{c}{\lambda' L^2}\Big)^{-L} = G(\lambda', L). \tag{4.1}$$

In perturbation theory around $\lambda = 0$ we find that the function vanishes at L leading loop orders

$$F(\lambda, L) = \sum_{\ell=0}^{\infty} F_\ell\, \lambda^\ell = \frac{\lambda^L}{c^L} - \frac{\lambda^{L+1}}{c^{L+1}} + \frac{\lambda^{L+2}}{c^{L+2}} + \cdots\,, \qquad \text{i.e.} \quad F_\ell(L) = 0 \quad \text{for} \quad \ell < L \tag{4.2}$$

The factor λ^L mimics the effect of wrapping interactions in gauge theory as explained below. When we now go to the thermodynamic limit $L \to \infty$, we see that all coefficients F_ℓ are zero.

Now let us take the thermodynamic limit first. The large L limit of $G(\lambda', L) = \sum_\ell G_\ell\, \lambda'^\ell$ yields $G(\lambda') = 1$ in a straightforward fashion. This result depends crucially on the function $F(\lambda, L)$. Currently, we do not know how to incorporate wrapping interactions, but λ^L alone would not have a sensible thermodynamic limit. To compensate this, we have introduced some function $1/(c+\lambda)^L$. Clearly we cannot currently prove that gauge theory produces a function like this, but it appears to be a definite possibility. In our toy example, the expansion in λ' gives $G_0 = 1$ and $G_\ell = 0$ otherwise.

In conclusion we find $G_\ell = \delta_{\ell,0}$ while $F_0 = 0$ which demonstrates the noncommutativity of the diagram in figure 1 in an example potentially relevant to our context. It is not hard to construct a function $F(\lambda, L)$ which yields arbitrary coefficients G_ℓ while all F_ℓ remain zero.

Note however that there is a sign of the non-commutativity in (4.2): A correct scaling behavior would require the coefficient F_ℓ to scale as $L^{-2\ell}$. In particular for $\ell = L$, the coefficient should scale as L^{-2L} instead of c^{-L}. Therefore one can say that the function F violates the scaling law at weak coupling, but in a mild way that is easily overlooked. This parallels observations made for the Inozemtsev spin chain for which scaling is manifestly violated at weak coupling. Nevertheless, when one does not expand for small λ scaling might be recovered [18].

4.3 Wrapping interactions

In perturbative field theory, the contributions to the dilatation operator are derived from Feynman diagrams. Let us consider a local operator with finitely many fields. At lower loop orders ℓ a planar Feynman diagram attaches to a number of neighboring sites along the spin chain. When the loop order increases, this region stretches until it wraps completely around the trace. At this point, when $\ell \geq L$, our methods cease to work: we know there are further contributions which couple only to states of a fixed length, but we currently have no information about their structure. Furthermore, it is not quite clear how to achieve a BMN limit or integrability.

Something very similar is true for long-range spin chains, and their solution by the *asymptotic* Bethe ansatz. It is again useful to take inspiration from the Inozemtsev model [19, 20]. There the second charge (i.e. the hamiltonian) is L dependent, and the strength of the spin-spin interactions is governed by a doubly-periodic elliptic Weierstrass function. Its imaginary period is related to the coupling constant, and its real period to L. In the asymptotic limit $L \to \infty$ the real period disappears and the interaction strength turns into a hyperbolic function. It is precisely this reduced model which is properly described by the asymptotic Bethe ansatz, as in (2.14). This ansatz actually works even better than one might expect at first sight: It does not strictly require $L = \infty$, but only that $\ell < L$, where ℓ measures the interaction range, as in field theory.

Our novel long-range spin chain is clearly very closely related to the Inozemtsev model. We expect that its hamiltonian, along with all other charges, also has a "periodized" extension, in full analogy with going from the hyperbolic to the elliptic Inozemtsev interaction. Hopefully this extension will still be consistent with our construction principles spelled out at the beginning of section 2, namely (i) integrability, (ii) compatibility with field theory, and (iii) BMN scaling. If we are lucky, the full model will still be *unique*, and should then correspond to the non-perturbative planar dilatation operator of $\mathcal{N} = 4$ theory.

While it is very reasonable to assume that this hamiltonian exists, it is currently unclear whether it may be explicitly written down in any useful fashion. Luckily, our results above suggest that this might not be necessary or even desirable, if we succeed in properly including the effect of wrappings into the Bethe ansatz. This is however not an easy problem, which has not yet been solved for the Inozemtsev model either [20].

We cannot currently offer a quantitative theory of wrapping effects, and are thus unable to explain why they only modify the thermodynamic limit starting at $\mathcal{O}\left(\lambda'\right)^3$. Nevertheless, on a qualitative level much can be said in favor of the proposal that their inclusion will lead to a reconciliation of the current disagreements between string and gauge theory. First of all, it is reasonable to expect that wrappings should not affect the energy formula of the strict BMN limit [21]. This formula is obtained, from the point of view of our long-range Bethe ansatz, in a rather trivial fashion by neglecting magnon scattering altogether! In this "dilute gas" approximation one sets the right hand side of the Bethe equations (2.14) to 1. In contrast, the near BMN limit takes into account finite size $1/J$ corrections, which *are* including the scattering effects, as may be seen by inspecting the calculations of section 2.6. Clearly wrapping effects should be included into such finite size corrections when we scale according to the north-east corner of figure 1, but not when we scale as in the south-west corner of that diagram. As for spinning strings and the FT proposal, it is clear that magnon scattering is heavily influencing the computation of the energy (and all other charges). In fact, the Hilbert kernel of the relevant singular integral equations is precisely describing the local, pairwise interaction of a macroscopically large number of excitations.

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Therefore, as for near BMN, the two different scaling procedures of figure 1, which in- or exclude wrappings, are expected to influence the scattering phase shifts, and therefore the distribution density of magnon momenta. Wrappings are *not* expected to change the *form* of the contributions (i.e. their functional dependence on the magnon momentum and the coupling constant) of individual magnons to the overall expectation values of charges, which is precisely what we have been finding in section 3!

On a more technical level, we suspect that the unknown terms in the transfer matrix eigenvalues (2.26) will contribute when we scale according to the north-east corner of figure 1. It may be expected that a non-asymptotic Bethe ansatz (if it exists at all) will follow from the full analytic structure of the complete transfer matrix, just as in the one-loop case. Likewise, one would hope that the latter will also generate the correct expressions for the global charges as functions of the individual magnon contributions.

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A. The planar SYM spectrum

In this appendix we compare the planar gauge theory spectrum, which was obtained using the hamiltonian approach described in section 2.1, to the results of the long-range Bethe ansatz for small excitation numbers.

A.1 Lowest-lying states

In the following we will describe how to obtain results using the spin chain hamiltonian as well as the long-range Bethe ansatz in general. In the subsequent sections will be go into details for certain classes of states.

We start by computing the eigenvalues of the first few commuting charges in perturbative gauge theory. To obtain a matrix representation for the operators, we have applied the hamiltonian $\mathbf{H} \equiv \mathbf{Q}_2$. Here are the spin chain hamiltonian up to five-loops, $\mathcal{O}\left(g^8\right)$. The constants $\alpha, \beta_{1,2,3}$ do not influence the spectrum.[22])

$$\begin{aligned}
\mathbf{H}_0 &= \{\} - \{1\}, \\
\mathbf{H}_2 &= -2\{\} + 3\{1\} - \frac{1}{2}\left(\{1,2\} + \{2,1\}\right), \\
\mathbf{H}_4 &= \frac{15}{2}\{\} - 13\{1\} + \frac{1}{2}\{1,3\} +
\end{aligned}$$

[22]The four-loop and five-loop contributions contain some undetermined constants $\alpha, \beta_{1,2,3}$. These are unphysical and correspond to perturbative rotations of the space of states. They change the (unphysical) eigenstates, but not the (physical) eigenvalues and thus cannot be fixed (unless one finds a canonical way to write the higher order hamiltonians).

$$
\begin{aligned}
& +3\left(\{1,2\}+\{2,1\}\right)-\frac{1}{2}\left(\{1,2,3\}+\{3,2,1\}\right). \\
\mathbf{H}_6 \;=\; & -35\{\}+(67+4\alpha)\{1\}+\left(-\frac{21}{4}-2\alpha\right)\{1,3\}-\frac{1}{4}\{1,4\}+ \\
& +\left(\frac{151}{8}-4\alpha\right)\left(\{1,2\}+\{2,1\}\right)+2\alpha\left(\{1,3,2\}+\{2,1,3\}\right)+ \\
& +\frac{1}{4}\left(\{1,2,4\}+\{1,3,4\}+\{1,4,3\}+\{2,1,4\}\right)+(6+2\alpha)\left(\{1,2,3\}+\{3,2,1\}\right)+ \\
& +\left(-\frac{3}{4}-2\alpha\right)\{2,1,3,2\}+\left(\frac{9}{8}+2\alpha\right)\left(\{1,3,2,4\}+\{2,1,4,3\}\right)+ \\
& +\left(-\frac{1}{2}-\alpha\right)\left(\{1,2,4,3\}+\{1,4,3,2\}+\{2,1,3,4\}+\{3,2,1,4\}\right)- \\
& -\frac{5}{8}\left(\{1,2,3,4\}+\{4,3,2,1\}\right), \\
\mathbf{H}_8 \;=\; & +\left(\frac{1479}{8}\right)\{\}+\left(-\frac{1043}{4}-12\alpha-2\beta_1-4\beta_2\right)\{1\}+ \\
& +(-19+8\alpha-2\beta_1-4\beta_2)\{1,3\}+(5+2\alpha+4\beta_2+4\beta_3)\{1,4\}+\frac{1}{8}\{1,5\}+ \\
& +(11\alpha-4\beta_1+2\beta_3)\left(\{1,2\}+\{2,1\}\right)-\frac{1}{4}\{1,3,5\}+ \\
& +\left(\frac{251}{4}-5\alpha+2\beta_1-2\beta_3\right)\left(\{1,3,2\}+\{2,1,3\}\right)+ \\
& +(-3-\alpha-2\beta_3)\left(\{1,2,4\}+\{1,3,4\}+\{1,4,3\}+\{2,1,4\}\right)- \\
& -\frac{1}{8}\left(\{1,2,5\}+\{1,4,5\}+\{1,5,4\}+\{2,1,5\}\right)+ \\
& +\left(\frac{41}{4}-6\alpha+2\beta_1-4\beta_3\right)\left(\{1,2,3\}+\{3,2,1\}\right)+ \\
& +\left(-\frac{107}{2}+4\alpha-2\beta_1\right)\{2,1,3,2\}+ \\
& +\left(\frac{1}{4}+\beta_2\right)\left(\{1,3,2,5\}+\{1,3,5,4\}+\{1,4,3,5\}+\{2,1,3,5\}\right)+ \\
& +\left(\frac{183}{4}-6\alpha+2\beta_1-2\beta_2\right)\left(\{1,3,2,4\}+\{2,1,4,3\}\right)+ \\
& +\left(-\frac{3}{4}-2\beta_2\right)\left(\{1,2,5,4\}+\{2,1,4,5\}\right)+(1+2\beta_2)\left(\{1,2,4,5\}+\{2,1,5,4\}\right)+ \\
& +\left(-\frac{51}{2}+\frac{5}{2}\alpha-\beta_1+\beta_2+3\beta_3\right)\left(\{1,2,4,3\}+\{1,4,3,2\}+\{2,1,3,4\}+\{3,2,1,4\}\right)+ \\
& -\beta_2\left(\{1,2,3,5\}+\{1,3,4,5\}+\{1,5,4,3\}+\{3,2,1,5\}\right)+ \\
& +\left(\frac{35}{4}+\alpha+2\beta_3\right)\left(\{1,2,3,4\}+\{4,3,2,1\}\right)+ \\
& +\left(-\frac{7}{8}-\alpha+2\beta_3\right)\left(\{1,4,3,2,5\}+\{2,1,3,5,4\}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{1}{2}+\alpha\right)(\{1,3,2,5,4\}+\{2,1,4,3,5\})+\\
&+\left(\frac{5}{8}+\frac{1}{2}\alpha-\beta_3\right)(\{1,3,2,4,3\}+\{2,1,3,2,4\}+\{2,1,4,3,2\}+\{3,2,1,4,3\})+\\
&+\left(\frac{1}{4}-2\beta_3\right)(\{1,2,5,4,3\}+\{3,2,1,4,5\})+\\
&+\left(\frac{1}{4}+\frac{1}{2}\alpha+\beta_3\right)(\{1,2,4,3,5\}+\{1,3,2,4,5\}+\{2,1,5,4,3\}+\{3,2,1,5,4\})+\\
&+\left(-\frac{1}{2}\alpha-\beta_3\right)(\{1,2,3,5,4\}+\{1,5,4,3,2\}+\{2,1,3,4,5\}+\{4,3,2,1,5\})-\\
&-\frac{7}{8}(\{1,2,3,4,5\}+\{5,4,3,2,1\})\ .
\end{aligned}
\tag{A.1}
$$

We have applied also the charges $\mathbf{Q}_3, \mathbf{Q}_4$ (up to four loops, see [53]) and $\mathbf{Q}_5, \mathbf{Q}_6$ (up to two loops) to all states with a given length L and number of excitations M. The computations were performed using a set of `Mathematica` routines which will be given in [53]. Then, the leading order energy matrix was diagonalized in order to obtain the leading order energy eigenvalues. Next, the off-diagonal terms at higher-loops were removed iteratively by a sequence of similarity transformations. Afterwards, the hamiltonian is diagonal and we can read off the energy eigenvalues. The same similarity transformations which were used to make $\mathbf{Q}_2$ diagonal also diagonalize $\mathbf{Q}_{3,4,5,6}$ and we may read off their eigenvalues.

We present our findings for $\mathbf{Q}_2$, $\mathbf{Q}_3$ and $\mathbf{Q}_4$ up to $L = 8$ in table 1 (we omit the protected states with $M = 0$) which is read as follows. For each state there is a polynomial and we write down its coefficients up to $\mathcal{O}\left(g^8\right)$ and $\mathcal{O}\left(x^2\right)$. For single states the polynomial $P(x, g)$ equals simply

$$
P(x, g) = \mathbf{Q}_2(g) + x^2\, \mathbf{Q}_4(g). \tag{A.2}
$$

If there is more than one state transforming in the same representation, the eigenvalues are solutions to algebraic equations. These could be solved numerically, here we prefer to state the exact polynomial $P(\psi, x, g)$ of degree $k-1$ in ψ. The energy and charge eigenvalues are determined by the formula

$$
\psi = \mathbf{Q}_2(g) + x\, \mathbf{Q}_3(g) + x^2\, \mathbf{Q}_4(g) + \cdots, \qquad \psi^k = P(\psi, g, x). \tag{A.3}
$$

At first sight the terms linear in x may appear wrong and the corresponding charge $\mathbf{Q}_3(g)$ would have to be zero. For unpaired states with non-degenerate $\mathbf{Q}_2(g)$ this is true, but not so for pairs of degenerate states. Then the solution of the algebraic equation leads to terms of the sort $\sqrt{0 + x^2} = \pm x$, where the 0 symbolizes the degeneracy. For some states the interaction is longer than the state. In such a case, indicated by $*$ in the table, we do not know the energy/charge eigenvalue, see section 4.3.

Before we turn to comparing the Bethe ansatz to the hamiltonian method for a number of specific examples, let us briefly describe how the energy/charge eigenvalues are obtained from the Bethe equations in general. As we are interested in the higher loop corrections to these eigenvalues, we expand the Bethe roots φ_k in the coupling:

$$
\varphi_k = \varphi_{k,0} + g^2 \varphi_{k,2} + g^4 \varphi_{k,4} + \cdots . \tag{A.4}
$$

L	M	$\mathfrak{p}$	g^0x^0	g^2x^0	g^4x^0	g^6x^0	g^8x^0	g^0x^2	g^2x^2	g^4x^2	g^6x^2
4	2	$+$	$+6$	-12	$+42$	$*$	$*$	$+0$	$*$	$*$	$*$
5	2	$-$	$+4$	-6	$+17$	$-\frac{115}{2}$	$*$	$+\frac{8}{3}$	-8	$*$	$*$
6	2	$+$	$+10\psi$	-17ψ	$+\frac{117}{2}\psi$	$-\frac{1037}{4}\psi$	$+\frac{10525}{8}\psi$	$-\frac{10}{3}\psi$	$+30\psi$	$-\frac{381}{2}\psi$	$*$
			-20	$+60$	-230	$+1025$	$-\frac{10165}{2}$	$+0$	$-\frac{140}{3}$	$+420$	$*$
6	3	$-$	$+6$	-9	$+\frac{63}{2}$	$-\frac{621}{4}$	$+\frac{7047}{8}$	-6	$+36$	$-\frac{405}{2}$	$*$
7	2	$-$	$+2$	$-\frac{3}{2}$	$+\frac{37}{16}$	$-\frac{283}{64}$	$+\frac{9597}{1024}$	$+\frac{4}{3}$	$-\frac{5}{2}$	$+\frac{81}{16}$	$-\frac{707}{64}$
7	2	$-$	$+6$	$-\frac{21}{2}$	$+\frac{555}{16}$	$-\frac{8997}{64}$	$+\frac{651651}{1024}$	$+0$	$+\frac{9}{2}$	$-\frac{513}{16}$	$+\frac{11907}{64}$
7	3	$\pm$	$+10\psi$	-15ψ	$+50\psi$	$-\frac{875}{4}\psi$	$+\frac{4365}{4}\psi$	$-\frac{10}{3}\psi$	$+25\psi$	$-\frac{285}{2}\psi$	$+\frac{1615}{2}\psi$
			-25	$+75$	$-\frac{1225}{4}$	$+\frac{5875}{4}$	$-\frac{61775}{8}$	$+\frac{245}{12}$	-180	$+\frac{28145}{24}$	$-\frac{86875}{12}$
8	2	$+$	$+14\psi^2$	$-23\psi^2$	$+79\psi^2$	$-349\psi^2$	$+\frac{3527}{2}\psi^2$	$-\frac{14}{3}\psi^2$	$+39\psi^2$	$-250\psi^2$	$+\frac{4691}{3}\psi^2$
			-56ψ	$+172\psi$	-695ψ	$+3254\psi$	-16746ψ	$+\frac{56}{3}\psi$	$-\frac{700}{3}\psi$	$+\frac{5258}{3}\psi$	-11822ψ
			$+56$	-224	$+966$	-4585	$+23555$	$+0$	$+168$	$-\frac{5054}{3}$	$+\frac{38269}{3}$
8	3	$-$	$+6$	-9	$+33$	-162	$+\frac{1803}{2}$	-6	$+33$	-192	$+1191$
8	3	$\pm$	$+8\psi$	-10ψ	$+28\psi$	-102ψ	$+422\psi$	$+\frac{4}{3}\psi$	-2ψ	$+4\psi$	$-\frac{26}{3}\psi$
			-16	$+40$	-137	$+548$	-2394	$-\frac{4}{3}$	$-\frac{40}{3}$	$+\frac{328}{3}$	$-\frac{1948}{3}$
8	4	$+$	$+20\psi^2$	$-32\psi^2$	$+112\psi^2$	$-511\psi^2$	$+2665\psi^2$	$-\frac{32}{3}\psi^2$	$+72\psi^2$	$-442\psi^2$	$+\frac{8264}{3}\psi^2$
			-116ψ	$+340\psi$	-1400ψ	$+6938\psi$	-38244ψ	$+\frac{392}{3}\psi$	$-\frac{3100}{3}\psi$	$+\frac{20708}{3}\psi$	-45348ψ
			$+200$	-800	$+3600$	-18400	$+102950$	-320	$+2800$	$-\frac{58400}{3}$	$+\frac{389680}{3}$

Table 1: Five-loop energies and four-loop eigenvalues of the charges $\mathbf{Q}_{3,4}$. Please refer to appendix A.1 for an explanation.

This is inserted into the Bethe equations (2.30) where both, the left and the right hand side, are expanded in g^2. The zeroth order of this expansion is just the one-loop Bethe equation. These are relatively easy to solve for short spin chains with a small number of magnons. For increasing L and M, however, it becomes more and more difficult to find the solutions to these equations. In table 2 the one-loop roots of the states of table 1 are listed. Note that we have omitted the roots of the two-excitation states as the general formula for their momenta will be given in section A.2. The states marked * are mirror solutions and will be explained in section A.3. Instead of writing down the approximate numerical values of the Bethe roots, we prefer to give the exact algebraic equations whose roots, $\varphi_{k,0}$, are exactly the one-loop Bethe roots. In the case where there is more than one state (i.e. more than one set of Bethe roots) for a given L and M, we give one polynomial for all Bethe roots in all the different sets. It is left as an exercise for the reader to determine which root belongs to which state (or set).[23]

After having obtained the one-loop Bethe roots, solving the expanded Bethe equations order by order in g^2 for their higher-loop corrections becomes a purely algebraic exercise. Using these, the energy/charge eigenvalues are computed and subsequently compared to table 1.

[23] Here it is helpful to keep in mind some facts about how the Bethe roots of an $\mathfrak{su}(2)$ chain are distributed in the complex plain, see for example [5].

L	M	$\mathfrak{p}$	
6	4*	$-$	$48\varphi_0^4+72\varphi_0^2-1=0$
7	3	$\pm$	$960\varphi_0^6+80\varphi_0^4+180\varphi_0^2-9=0$
8	3	$\pm$	$16\varphi_0^6-8\varphi_0^4+9\varphi_0^2-1=0$
	4	$+$	$552960\varphi_0^{12}+460800\varphi_0^{10}-16128\varphi_0^8+81664\varphi_0^6-4464\varphi_0^4+648\varphi_0^2-1=0$
	6*	$-$	$64\varphi_0^6-208\varphi_0^4-308\varphi_0^2+1=0$

Table 2: One-loop Bethe roots.

In this context let us point out the importance of paired and unpaired states. The unpaired states correspond to symmetric distributions of the Bethe roots, $\{\varphi_k\}=\{-\varphi_k\}$, which in turn implies the vanishing of all odd charges. The momentum constraint $\mathbf{U}=1$ (2.27) is almost automatically satisfied. It merely implies that for odd M, in addition to one root at the origin $\varphi_1=0$, two of the roots must be at the singular points $\varphi_{2,3}\approx\pm\frac{i}{2}$ (cf. [26]). For even M there can be no such roots. This symmetry vastly simplifies the computation of the Bethe roots; we need to solve only half as many equations! For paired states this simplification does not apply and finding the roots is a formidable problem even for smaller values of (L,M).

A.2 Two excitations

Now that both the hamiltonian approach and the perturbative Bethe ansatz have been described in detail, we may compare the results that are obtained using these two procedures. Let us start by analyzing the states with two magnons [58]. On the perturbative gauge theory side of our discussion one can extend the conjectured all-loop formula (6) in [17]

$$\mathbf{D}(J,n,g)=J+2+\sum_{\ell=1}^{\infty}\left(8g^2\sin^2\frac{\pi n}{J+1}\right)^{\ell}\left(c_\ell+\sum_{k,h=1}^{\ell-1}c_{\ell,k,h}\frac{\cos^{2h}\frac{\pi n}{J+1}}{(J+1)^k}\right)\tag{A.5}$$

by matching more and more coefficients c_ℓ, $c_{\ell,k,h}$ to sufficiently many two-excitation states. We present a summary of findings for the coefficients for the two-excitation formula (A.5).

$$\begin{aligned}
c_1&=+1\,,\\
c_2&=-\frac{1}{4}\,, & c_{2,1,1}&=-1\,,\\
c_3&=+\frac{1}{8}\,, & c_{3,k,h}&=\begin{pmatrix}+\frac{3}{4}&+\frac{1}{2}\\-\frac{3}{4}&+\frac{5}{2}\end{pmatrix},\\
c_4&=-\frac{5}{64}\,, & c_{4,k,h}&=\begin{pmatrix}-\frac{5}{8}&-\frac{5}{12}&-\frac{1}{3}\\+\frac{3}{4}&-\frac{7}{4}&-\frac{7}{2}\\-\frac{1}{2}&+\frac{59}{12}&-\frac{49}{6}\end{pmatrix},\\
c_5&=+\frac{7}{128}\,, & c_{5,k,h}&=\begin{pmatrix}+\frac{35}{64}&+\frac{35}{96}&+\frac{7}{24}&+\frac{1}{4}\\-\frac{45}{64}&+\frac{185}{96}&+\frac{131}{48}&+\frac{33}{8}\\+\frac{5}{8}&-\frac{125}{24}&-\frac{13}{24}&+\frac{81}{4}\\-\frac{5}{16}&+\frac{305}{48}&-\frac{1319}{48}&+\frac{243}{8}\end{pmatrix}.
\end{aligned}\tag{A.6}$$

When the coefficients have been determined, we may compare the formula to the results of the Bethe equations. As mentioned above, the states with two magnons turn out to be unpaired due to momentum conservation, i.e. we only have to solve one Bethe equation (2.30) for $L = J + 2$ in $\varphi \equiv \varphi_1 = -\varphi_2$:

$$\left(\frac{\varphi - \frac{i}{2}}{\varphi + \frac{i}{2}}\right)^{J+1} = \left(\frac{1 + \sqrt{1 - 2g^2/(\varphi + \frac{i}{2})^2}}{1 + \sqrt{1 - 2g^2/(\varphi - \frac{i}{2})^2}}\right)^{J+2} . \tag{A.7}$$

Solving this Bethe equation order by order in g^2 in the way described above leads to

$$\varphi = \frac{1}{2}\cot\frac{\pi n}{J+1}\Bigg[1 + 4g^2\frac{J+2}{J+1}\sin^2\frac{\pi n}{J+1} - \\ - 2g^4\frac{(J+2)(J-1+6\cos^2\frac{\pi n}{J+1})}{(J+1)^2}\sin^4\frac{n\pi}{J+1} + \mathcal{O}\left(g^6\right)\Bigg] . \tag{A.8}$$

The higher order terms are rather lengthy which is why we do not explicitly write them down here. After plugging this into the energy formula (2.20) together with (2.23), we obtain (A.5) for the first few loop-orders.

At this point let us also say a few words about the inhomogeneous Bethe equations (2.42). The procedure of computing the Bethe roots is exactly the same as before, only the left hand side of (2.42) differs from (2.30). It is possible to make this replacement since both equations agree up to $\mathcal{O}\left(g^{2L}\right)$. However, this is also the order at which the contributions of wrapping interactions have to be taken into account. Since we do not know how to do this, both types of Bethe equations are equivalent at the desired accuracy. The benefit of the inhomogeneous Bethe equations will be demonstrated in the following section. As an example let us calculate the Bethe roots of the Konishi descendant $(L, M) = (4, 2)$. These can be solved for exactly, we find

$$\varphi_{1,2} = \pm\sqrt{-\frac{1}{12} + \frac{1}{3}g^2 + \frac{1}{6}\sqrt{1 + 4g^2 + 10g^4}} . \tag{A.9}$$

The corresponding exact inhomogeneous transfer matrix is

$$\bar{\mathbf{T}}(\varphi) = \frac{5}{8} - g^2 + g^4 + \sqrt{1 + 4g^2 + 10g^4} + 3\varphi^2 - 4g^2\varphi^2 + 2\varphi^4 . \tag{A.10}$$

As expected, the resulting energy eigenvalue agrees with (A.5) up to and including $\mathcal{O}\left(g^6\right)$.

A.3 Singular solutions

Next, let us analyze unpaired three-excitation states [12]

$$\sum_{p=1}^{L-4}(-1)^p \operatorname{Tr}\phi\mathcal{Z}^p\phi\mathcal{Z}^{L-3-p}\phi + \mathcal{O}\left(g^2\right) \tag{A.11}$$

at higher loops using perturbative gauge theory techniques. Note that this exact one-loop form of the eigenstates is corrected at higher-loops. We find for the scaling dimensions

$$\mathbf{D} = 2 ,$$

$$
\begin{aligned}
\mathbf{D} &= 4+6g^2-12g^4+\frac{84}{2}g^6+\cdots\,,\\
\mathbf{D} &= 6+6g^2-9g^4+\frac{63}{2}g^6-\frac{621}{4}g^8+\frac{7047}{8}g^{10}+\cdots\,,\\
\mathbf{D} &= 8+6g^2-9g^4+\frac{66}{2}g^6-\frac{648}{4}g^8+\frac{7212}{8}g^{10}+\cdots\,,\\
\mathbf{D} &= 10+6g^2-9g^4+\frac{66}{2}g^6-\frac{645}{4}g^8+\frac{7179}{8}g^{10}+\cdots\,,\\
\mathbf{D} &= 12+6g^2-9g^4+\frac{66}{2}g^6-\frac{645}{4}g^8+\frac{7182}{8}g^{10}+\cdots\,,\\
\mathbf{D} &= 14+6g^2-9g^4+\frac{66}{2}g^6-\frac{645}{4}g^8+\frac{7182}{8}g^{10}+\cdots\,,\\
&\quad\ldots\,,
\end{aligned}
\tag{A.12}
$$

where we have added the dimension-two half-BPS state and the above Konishi descendant which appear to be the natural first two elements of this sequence.

We observe that all corrections $\mathbf{D}_k$ to the scaling dimensions below the "diagonal" $k \leq L-2$, are equal. Incidentally these coefficients agree with the formula

$$
\mathbf{D}(g) = L + \left(\sqrt{1+8g^2}-1\right) + \left(\sqrt{1+2g^2}-1\right) + \left(\sqrt{1+2g^2}-1\right). \tag{A.13}
$$

We may interpret the three terms in parentheses as the energies of the three excitations. Then this form can be taken as a clear confirmation of an integrable system with elastic scattering of excitations.

Only if the loop order is at least half the classical dimension, the pattern breaks down. Interestingly, if the loop order is exactly half the classical dimension, the coefficient is decreased by $3 \cdot 2^{2-\ell}$. It would be of great importance to understand the changes further away from the diagonal. This might provide us with clues about wrapping interactions, which, in the above example, obscure the scaling dimension of the Konishi state beyond three-loops.

For completeness, we state an analogous all-loop conjecture for the higher charges

$$
\mathbf{Q}_r = \frac{(+i)^{r-2}+(-i)^{r-2}}{(r-1)g^{2r-2}} \left(2^{1-r}\left(\sqrt{1+8g^2}-1\right)^{r-1} + \left(\sqrt{1+2g^2}-1\right)^{r-1}\right). \tag{A.14}
$$

Alternatively, in terms of a transfer matrix:

$$
\mathbf{T}(x) = \frac{x-\frac{i}{4}(1+\sqrt{1+8g^2})}{x+\frac{i}{4}(1+\sqrt{1+8g^2})}\,\frac{x-\frac{i}{2}(1+\sqrt{1+2g^2})}{x+\frac{i}{2}(1+\sqrt{1+2g^2})} \tag{A.15}
$$

Now let us discuss how this result can be reproduced using the Bethe ansatz. The unpaired three-excitation states are singular solutions of the Bethe equations.[24] At leading order, the Bethe roots are at $\varphi_1 = 0$ and the singular points $\varphi_{2,3} = \pm\frac{i}{2}$, see e.g. [26]. The singular roots lead to divergencies in the Bethe equations which would have to be regularized. While it is not clear to us how the regularization can be done at higher loops, there is an alternative way to solve the equations without the need to regularize. The Bethe

[24]This is related to the fact that in the state (A.11) two of the fields ϕ are *always* next to each other.

equations follow from the requirement that $\bar{\mathbf{T}}(\varphi)$ must not have poles at $\varphi = \varphi_k$. Here we are forced to use the transfer matrix of the inhomogeneous Bethe ansatz in section 2.4 and not the one of section 2.3. The reason is that the function $x(\varphi \pm \frac{i}{2})^L$ introduces additional overlapping singularities at $\varphi = \pm\frac{i}{2}$, while the polynomial $P_L(\varphi \pm \frac{i}{2})$ certainly does not. Therefore, only the inhomogeneous Bethe equations can be used to find the quantum corrections to the singular Bethe roots. Interestingly, one finds their positions not to be modified up to $\mathcal{O}\left(g^L\right)$ and (A.15) follows straightforwardly from (2.31)

$$\mathbf{T}(x) = \frac{x - x(0+\frac{i}{2})}{x - x(0-\frac{i}{2})}\,\frac{x - x(+\frac{i}{2}+\frac{i}{2})}{x - x(+\frac{i}{2}-\frac{i}{2})}\,\frac{x - x(-\frac{i}{2}+\frac{i}{2})}{x - x(-\frac{i}{2}-\frac{i}{2})} + \cdots = \frac{x - x(+\frac{i}{2})}{x - x(-\frac{i}{2})}\,\frac{x - x(+i)}{x - x(-i)} + \cdots \tag{A.16}$$

When the shifts of the poles at $\mathcal{O}\left(g^L\right)$ are properly taken into account, one finally obtains the corrections above the diagonal in (A.12).

As an example let us consider the state $(L, M) = (4, 3)$, which is the mirror state of the Konishi descendant we computed in the previous subsection. The mirror of a state (L, M) is a (zero-norm) state of the type $(L, L - M + 1)$ which has the precisely the same charges. Its Bethe roots can be determined analytically

$$\varphi_1 = 0\,, \qquad \varphi_{2,3} = \pm\sqrt{-\frac{3}{4} - g^2 + \frac{1}{2}\sqrt{1 + 4g^2 + 10g^4}}\,. \tag{A.17}$$

The exact transfer matrix $\bar{\mathbf{T}}(\varphi)$ is the same as for the $(4, 2)$ state (A.10). This demonstrates that the equations in section 2.4 are fully consistent when computing the unphysical transfer matrix $\bar{\mathbf{T}}(\varphi)$.

For the physical charges $\mathbf{Q}_r$ the situation is slightly different. The charges do agree with the ones of the Konishi descendant.[25] As expected, this agreement persists only for the first few orders.In particular, $\mathbf{Q}_2$ agrees up to $\mathcal{O}\left(g^4\right)$ and $\mathbf{Q}_4$ up to $\mathcal{O}\left(g^0\right)$. Remarkably, this is also precisely the accuracy at which wrappings occur, which constitutes some evidence for the improper treatment of wrapping interactions by our ansatz. We therefore conclude, that the ansatz of section 2.4 yields self-consistent results for the physical charges only for low loop orders.

In this example we considered the mirror of a regular state in order to investigate a singular state. However, it is also possible to reverse the line of argumentation: When interested in a singular three-excitation state, one can instead consider its mirror, which is a regular, unpaired $(L - 2)$-excitation state. The procedure is the same as for ordinary unpaired states and we will not discuss it in any more detail. Instead we refer the reader to table 2 where the one-loop roots of some mirror states are listed (marked by *). The fact that these states have the same energy eigenvalues (up to "wrapping order") as the original, singular states with three excitations shows that the inhomogeneous Bethe ansatz is consistent.

[25] The physical charges of the singular solutions need to be regulated. Here, g acts as a natural regulator, when corrections to the roots are taken into account. This leaves some spurious terms of the sort $\sqrt{g}$ in the charge eigenvalues which we shall ignore.

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A.4 Paired three-excitation states

For three excitations there exist also paired states. Finding the leading order Bethe roots for these states is more complicated as we cannot use the symmetry argument $\varphi_{2k} = -\varphi_{2k+1}$. Therefore, one has to work with the whole set of Bethe equations. The longer the chain and the more excitations it has, the more difficult it becomes to solve the equations. For the states $(L, M) = (7, 3)$ and $(L, M) = (8, 3)$ we have used the resultant of polynomials in several variables in order to iteratively remove their dependence on all but a single variable.[26] The results are given by the corresponding equations in table 2. Each of these equations is solved by two sets of momenta $\{\varphi_k\}$ and $\{\varphi'_k\} = \{-\varphi_k\}$. Note the specific distribution of the three momenta in each set: one momentum lies on the positive (negative) real axis while the other two momenta are related by complex conjugation and have a negative (positive) real part. The higher even charges of the two sets are equal ($\mathbf{Q}'_{2r} = \mathbf{Q}_{2r}$) whereas the odd ones differ by an overall minus sign ($\mathbf{Q}'_{2r+1} = -\mathbf{Q}_{2r+1}$). We find agreement with the eigenvalues of the spin chain charges.

A.5 Higher excitations

For all states with an even number of magnons (and $M \geq 4$) the procedure is exactly the same as for the paired and unpaired states described above. The only complication arises when trying to solve the one-loop Bethe ansatz.

For the unpaired states with an odd number of excitations one knows that three of the one-loop roots are *singular*, i.e. their positions are $\varphi_1 = 0$, $\varphi_{2,3} = \pm\frac{i}{2}$. The remaining roots are again symmetric $\varphi_{2k} = -\varphi_{2k+1}$. As in the case of the unpaired three-excitation states we find that the singular roots do not receive corrections up to (and including) $\mathcal{O}\left(g^{L-2}\right)$ whereas the roots φ_k with $k \geq 4$ are corrected at every order in g^2.

We have specifically checked the agreement of results for the unpaired $(8, 4)$, $(9, 4)$, $(10, 4)$ and $(10, 5)$ states.

B. Inhomogeneous long-range spin chains

Our findings in section 2.4 that the novel spin chain may be interpreted as an *inhomogeneous* spin chain appears to be more generally true for long-range chains, as we will show in this appendix. In section 2.3 we have inverted the relation $\varphi(p)$ for our spin chain model as

$$\exp(ip) = \frac{x(\varphi + \frac{i}{2})}{x(\varphi - \frac{i}{2})} \tag{B.1}$$

and found a function $x(\varphi)$ so that the Bethe ansatz can be expressed as follows

$$\frac{x(\varphi_k + \frac{i}{2})^L}{x(\varphi_k - \frac{i}{2})^L} = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{\varphi_k - \varphi_j + i}{\varphi_k - \varphi_j - i}\,. \tag{B.2}$$

[26]The resultant R of two polynomials $P(x)$ and $P'(x)$ is zero if and only if they have a common root. When the polynomials also depend on further variables y_i, the vanishing of the resultant $R(y_i) = 0$ gives an algebraic constraint among the y_i's alone.

In section 2.4 we have then truncated the expansion of $x(\varphi)^L$ to a polynomial $P_L(\varphi)$ in order to relate the model to an inhomogeneous spin chain. Here we would like to repeat this exercise for more general long-range spin chains.

Let us start with the Inozemtsev spin chain [19, 20] as treated in [18]. The relation $\varphi(p)$ is given by (t is the coupling constant proportional to g^2)

$$\varphi(p) = \frac{1}{2}\cot\left(\frac{1}{2}p\right) + \sum_{n=1}^{\infty} \frac{4t^n \sin\left(\frac{1}{2}p\right)\cos\left(\frac{1}{2}p\right)}{(1-t^n)^2 + 4t^n \sin\left(\frac{1}{2}p\right)^2}\,. \tag{B.3}$$

The inversion of this relation is given by (B.1) and the function

$$x(\varphi) = \varphi - \frac{t}{\varphi} - \left(\frac{1}{\varphi^3} + \frac{3}{\varphi}\right)t^2 - \left(\frac{2}{\varphi^5} + \frac{7}{\varphi^3} + \frac{4}{\varphi}\right)t^3 - \left(\frac{5}{\varphi^7} + \frac{22}{\varphi^5} + \frac{20}{\varphi^3} + \frac{7}{\varphi}\right)t^4 - \cdots\,. \tag{B.4}$$

It is also useful to note its inverse

$$\varphi(x) = x + \frac{t}{x} + \frac{3t^2}{x} + \left(\frac{1}{x^3} + \frac{4}{x}\right)t^3 + \left(\frac{3}{x^3} + \frac{7}{x}\right)t^4 + \left(\frac{2}{x^5} + \frac{9}{x^3} + \frac{6}{x}\right)t^5 + \cdots\,. \tag{B.5}$$

It would be interesting to investigate also the physical charges or the transfer matrix. Possibly they are also given by (2.31), (2.32) or similar expressions involving $x(\varphi)$. Furthermore it would be interesting to find exact, analytic expressions for these functions.

The function (B.4) has a special property that allows us to reformulate the model as an inhomogeneous spin chain: The expansion of $x(\varphi)^L$ in powers of t up to $\mathcal{O}\left(t^{L-1}\right)$ is a polynomial in φ. Inverse powers of φ start contributing only at $\mathcal{O}\left(t^{L}\right)$. At this order, however, wrapping interaction start to contribute and the asymptotic Bethe ansatz does not apply anymore. Thus we may truncate $x(\varphi)^L$ at $\mathcal{O}\left(t^{L-1}\right)$ and get a polynomial $P_L(\varphi)$ of degree L, precisely what is needed for an inhomogeneous spin chain.

This property can be used to find functions $x(\varphi)$ for more general long-range spin chains. Here it is more useful to investigate the inverse $\varphi(x)$. We find that precisely the functions of the form (B.5), but with different coefficients, have this property. It is interesting to note that then the coefficients of t^n/φ^{2n-1} in $x(\varphi)$ are always the Catalan numbers[27] from the expansion of $\frac{1}{2}x + \frac{1}{2}x\sqrt{1-4ct/x^2}$. This is because these coefficients are determined by the first two terms in (B.5), $x + ct/x$, only. Remarkably, the restricted form (B.5) confirms the claimed uniqueness of our model specified by the conditions (i)–(iii) in section 2: For a correct scaling behavior, all terms should scale as L, i.e. only terms proportional to t^n/x^{2n-1}. The only allowed terms in (B.5) are x and t/x in agreement with our function (2.36).

C. Elliptic solutions of the string and gauge Bethe equations

In section 3 we compared the Bethe equations for semi-classical string theory and asymptotic gauge theory in generality. It is interesting to investigate the consequences for the

[27] The Catalan numbers have also been found in $\mathcal{N}=4$ SYM as the coefficients of the maximal shifts of a single excitation [56]. If directly related, this observation gives support to the conjecture of all-loop integrability [12].

analytic structure of the respective solutions on some explicit, solvable examples. In this appendix we will therefore study the string and gauge Bethe equations for the "folded" and the "circular" spinning string.[28] These are the simplest families of solutions which still depend on a continuous parameter, namely the filling fraction $\alpha = \frac{M}{L}$. As a byproduct we will verify that the classical sigma model Bethe equation of [35] indeed reproduces the energies of these spinning string configurations which were previously obtained by simpler, but less systematic methods [23, 24, 25]. We shall also find that the all-loop gauge solutions are significantly more complicated in analytic structure. The folded string is believed to correspond to the ground state of the representation carrying the charges M and $L - M$. The resulting two-cut solutions may be expressed through elliptic functions, and are closely related to the ones describing the multicritical $\mathrm{O}(\pm 2)$ matrix model [59]. A simplifying feature is that all odd charges are zero. These solutions were crucial in establishing for the first time the agreement of "long operator" anomalous dimensions and semi-classical string solutions at one-loop [26], at two loops [18], as well as the matching of integrable structures up to two loops [29, 30]. They also led to the discovery of the three-loop disagreement [18]. The calculations below are straightforward modifications of the ones presented in [26, 27, 29, 18] and we refer to these papers for further details.

Here we briefly state our conventions for the elliptic integrals appearing below. The complete elliptic integrals of the first (K) and second (E) kind are

$$\mathrm{K}(q) \equiv \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - q\sin^2\varphi}} \qquad\qquad \mathrm{E}(q) \equiv \int_0^{\pi/2} d\varphi\, \sqrt{1 - q\sin^2\varphi}\,. \tag{C.1}$$

and the complete elliptic integral of the third kind is

$$\Pi(m^2, q) \equiv \int_0^{\pi/2} \frac{d\varphi}{(1 - m^2\sin^2\varphi)\sqrt{1 - q\sin^2\varphi}}\,. \tag{C.2}$$

C.1 The folded string

C.1.1 Semi-classical string solution

Let us write down the classical string equations of section 3.2 for the case of exactly two contours $\mathcal{C}_+$ and $\mathcal{C}_-$ which are mutual images w.r.t. reflection around the imaginary axis. The Bethe equation (3.21) becomes

$$\fint_{\mathcal{C}_+} dx'\, \frac{\sigma(x')\, x}{x^2 - x'^2} = \frac{\mathcal{E}}{4}\, \frac{x}{x^2 - \frac{1}{2}g^2} + \frac{1}{2}\pi n_+ \qquad \text{with } x \in \mathcal{C}_+\,, \tag{C.3}$$

and the normalization condition (3.16) reads

$$\int_{\mathcal{C}_+} dx\, \sigma(x) \left(1 - \frac{g^2}{2x^2}\right) = \frac{\alpha}{2}\,. \tag{C.4}$$

[28] The below calculations in the case of the classical string sigma model Bethe equation were independently performed by G. Arutyunov (unpublished).

The resolvent (3.15) is a function analytic throughout the spectral x-plane, except for the cuts $\mathcal{C}_+$ and $\mathcal{C}_-$:

$$\mathcal{G}(x) = 2x \int_{\mathcal{C}_+} dx' \, \frac{\sigma(x')}{x'^2 - x^2} \,. \tag{C.5}$$

For small x we may expand the resolvent as a Taylor series in the local charges

$$\mathcal{G}(x) = \sum_{r=1}^{\infty} \mathcal{Q}_{2r} x^{2r-1} \qquad \text{with } \mathcal{Q}_{2r} = 2 \int_{\mathcal{C}_+} dx \, \frac{\sigma(x)}{x^{2r}} \,, \tag{C.6}$$

cf. (3.17), (3.18). Note that the odd charges are zero: $\mathcal{Q}_{2r-1} = 0$, and we recall the relation between the scaled string energy and the second charge (3.20): $\mathcal{E} = 1 + g^2 \mathcal{Q}_2(g)$. It is easily seen that the mode number n_+ may be absorbed, after rescaling the spectral parameter $x \mapsto x/n_+$, into the coupling constant $g \mapsto g/n_+$; we nevertheless keep full $n_+ = -n$ dependence.

It is technically convenient to solve this equation by analytically continuing to a negative filling fraction $\alpha < 0$, as in [26]. The complex cuts $\mathcal{C}_+$ and $\mathcal{C}_-$ are flipped to, respectively, the real intervals $[a, b]$ and $[-b, -a]$. This involves some sign changes in the above equations, which now read

$$⨍_a^b dx' \, \frac{\sigma(x') \, x}{x'^2 - x^2} = \frac{\mathcal{E}}{4} \frac{x}{x^2 - \frac{1}{2} g^2} - \frac{\pi n}{2} \qquad \text{with } \int_a^b dx \, \sigma(x) \left(1 - \frac{g^2}{2x^2}\right) = -\frac{\alpha}{2} \,, \tag{C.7}$$

and

$$\mathcal{G}(x) = 2x \int_a^b dx' \, \frac{\sigma(x')}{x^2 - x'^2} \,, \tag{C.8}$$

while (3.20) is unchanged. The solution of (C.7) could be obtained by an inverse Hilbert transform. However, in line with the connotation of the word "resolvent", it is technically easier to directly find an integral representation of $\mathcal{G}(x)$ with the required analytic properties and boundary conditions. One finds (here $q = 1 - a^2/b^2$) that (C.8) is explicitly given by

$$\begin{aligned} \mathcal{G}(x) \; = \; & -\frac{\mathcal{E}}{2} \frac{x}{x^2 - \frac{1}{2} g^2} + \frac{2na^2}{xb} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \, \Pi\left(-q \frac{x^2}{a^2 - x^2}, q\right) + \\ & + \frac{\mathcal{E}}{2x} \frac{\frac{1}{2} g^2}{x^2 - \frac{1}{2} g^2} \sqrt{\frac{(b^2 - x^2)(a^2 - x^2)}{(b^2 - \frac{1}{2} g^2)(a^2 - \frac{1}{2} g^2)}} \,, \end{aligned} \tag{C.9}$$

which is the form of $\mathcal{G}(x)$ appropriate for an expansion near $x = 0$, as needed for generating the local charges

$$\mathcal{G}(x) = \sum_{r=1}^{\infty} \mathcal{Q}_{2r} x^{2r-1} \qquad \text{with } \mathcal{Q}_{2r} = -2 \int_a^b dx \, \frac{\sigma(x)}{x^{2r}} \,. \tag{C.10}$$

We can now also read off the pseudodensity $\sigma(x)$ as the discontinuity of $\mathcal{G}(x)$ on the cut:

$$\sigma(x) = \frac{1}{2\pi x b} \sqrt{\frac{x^2 - a^2}{b^2 - x^2}} \left[b \, \mathcal{E} \, \frac{x^2}{x^2 - \frac{1}{2} g^2} \sqrt{\frac{b^2 - \frac{1}{2} g^2}{a^2 - \frac{1}{2} g^2}} - 4nx^2 \, \Pi\left(\frac{b^2 - x^2}{b^2}, q\right) \right] . \tag{C.11}$$

Furthermore, the known behavior of the resolvent at $x = 0$, namely $\mathcal{G}(x) = \frac{0}{x} + \mathcal{Q}_2 x + \mathcal{O}\left(x^3\right)$, yields two conditions:

$$\frac{\mathcal{E}}{\sqrt{(b^2 - \frac{1}{2}g^2)(a^2 - \frac{1}{2}g^2)}} = \frac{4n}{b}\,\mathrm{K}(q) \qquad \text{and} \qquad \mathrm{K}(q) = \frac{1}{4na} + \frac{g^2}{2a^2}\,\mathrm{E}(q)\,, \tag{C.12}$$

while a third condition is obtained from the behavior at $x \to \infty$, namely $\mathcal{G}(x) \to \frac{2}{x}\int_a^b dx'\sigma(x')$ (note that the last term in (C.9) flips sign for large values of x):

$$\alpha = \frac{1}{2} - 2nb\,\mathrm{E}(q) + \frac{ng^2}{b}\,\mathrm{K}(q)\,. \tag{C.13}$$

These three equations determine the three unknowns a, b and $\mathcal{E}$ as a function of the filling fraction α and the coupling constant g. One checks that they indeed reproduce the energy of the folded string as first obtained without Bethe ansatz in [24].

C.1.2 All-loop asymptotic gauge solution

In order to solve the singular integral equation (3.12) for perturbative gauge theory it is useful to introduce a φ-resolvent $\bar{\mathbf{G}}(\varphi)$ through

$$\bar{\mathbf{G}}(\varphi) = \int_{\mathbf{C}} \frac{d\varphi'\rho(\varphi')}{\varphi' - \varphi} = \sum_{r=1}^{\infty} \bar{\mathbf{Q}}_r\,\varphi^{r-1} \qquad \text{with } \bar{\mathbf{Q}}_r = \int_{\mathbf{C}} \frac{d\varphi\,\rho(\varphi)}{\varphi^r}\,. \tag{C.14}$$

Here (and in similar expressions for the remainder of this appendix) it is understood that, while $\bar{\mathbf{G}}(\varphi)$ is defined throughout the complex φ-plane, its expansion in local charges is only possible in a finite domain around $\varphi = 0$. Note however that this resolvent does *not* correspond to the scaling limit of the (logarithm of) the transfer matrix (2.26), except for the one-loop approximation. Accordingly, the proper gauge charges $\mathbf{Q}_r$ are not given by the moments $\bar{\mathbf{Q}}_r$ beyond one loop. Instead the former are linear combinations of the latter, cf. [18], as coded into the equation (3.9).

In our perturbative gauge theory ansatz, the two-cut Bethe equation of the folded string in the last section C.1.1 is replaced by

$$⨍_a^b \frac{d\varphi'\,\rho(\varphi')\,\varphi}{\varphi'^2 - \varphi^2} = \frac{1}{4}\frac{1}{\sqrt{\varphi^2 - 2g^2}} - \frac{\pi n}{2} \qquad \text{with } \int_a^b d\varphi\,\rho(\varphi) = -\frac{\alpha}{2}\,, \tag{C.15}$$

where we are using the same procedure of analytical continuation to negative filling α. (For notational simplicity we will again use the interval boundary values a, b even though they functionally differ between string and gauge theory.) The φ-resolvent becomes

$$\bar{\mathbf{G}}(\varphi) = 2\varphi\int_a^b \frac{d\varphi'\,\rho(\varphi')}{\varphi^2 - \varphi'^2} = \sum_{r=1}^{\infty} \bar{\mathbf{Q}}_{2r}\varphi^{2r-1} \qquad \text{with } \bar{\mathbf{Q}}_{2r} = -2\int_a^b \frac{d\varphi\,\rho(\varphi)}{\varphi^{2r}}\,. \tag{C.16}$$

The resolvent (C.16) is determined to be

$$\bar{\mathbf{G}}(\varphi) = \frac{2na^2}{\varphi b}\sqrt{\frac{b^2 - \varphi^2}{a^2 - \varphi^2}}\,\Pi\left(-q\,\frac{\varphi^2}{a^2 - \varphi^2}, q\right) -$$

$$-\frac{1}{\pi\varphi}\,\frac{b^2}{a\sqrt{b^2-2g^2}}\,\sqrt{\frac{a^2-\varphi^2}{b^2-\varphi^2}}\,\Pi\left(\frac{q}{1-q}\,\frac{\varphi^2}{b^2-\varphi^2},\frac{q}{1-q}\,\frac{2g^2}{b^2-2g^2}\right). \tag{C.17}$$

This representation is valid (without sign changes) both around $\varphi = 0$ and $\varphi = \infty$. The behavior of the resolvent at $\varphi = 0$, namely $\bar{\mathbf{G}}(\varphi) = \frac{0}{\varphi} + \mathcal{O}(\varphi)$, yields the condition

$$\mathrm{K}(q) = \frac{1}{2\pi n}\,\frac{b}{a\sqrt{b^2-2g^2}}\,\mathrm{K}\left(\frac{q}{1-q}\,\frac{2g^2}{b^2-2g^2}\right), \tag{C.18}$$

while a second condition is obtained from the behavior at $\varphi \to \infty$, namely $\bar{\mathbf{G}}(\varphi) \to -\frac{\alpha}{\varphi}$:

$$\alpha = -2nb\,\mathrm{E}(q) + \frac{1}{\pi}\,\frac{b^2}{a\sqrt{b^2-2g^2}}\,\Pi\left(\frac{-q}{1-q},\frac{q}{1-q}\,\frac{2g^2}{b^2-2g^2}\right). \tag{C.19}$$

These two equations determine the unknowns a, b as a function of the filling fraction α and the coupling constant g. The density $\rho(\varphi)$ is obtained as the discontinuity of the resolvent on the cut and reads

$$\begin{aligned}\rho(\varphi) \;=\; & \frac{1}{\pi^2\,\varphi}\,\frac{b^2}{a\sqrt{b^2-2g^2}}\,\sqrt{\frac{\varphi^2-a^2}{b^2-\varphi^2}}\,\Pi\left(\frac{b^2-\varphi^2}{b^2-2g^2}\,\frac{2g^2}{\varphi^2},\frac{q}{1-q}\,\frac{2g^2}{b^2-2g^2}\right) - \\ & -\frac{2n\varphi}{\pi b}\sqrt{\frac{\varphi^2-a^2}{b^2-\varphi^2}}\,\Pi\left(\frac{b^2-\varphi^2}{b^2},q\right).\end{aligned} \tag{C.20}$$

The φ-moments $\bar{\mathbf{Q}}_{2r}$ are now easily obtained explicitly by expanding (C.17). The proper gauge charges $\mathbf{Q}_{2r}$, however, still require further, unpleasant integrations, using (3.9), (3.6), which we have not been able to perform explicitly. E.g. the energy is obtained from the density (C.20) as the integral

$$\mathbf{D}(g) = 1 - \alpha - 2\int_a^b d\varphi\,\rho(\varphi)\,\frac{\varphi}{\sqrt{\varphi^2-2g^2}}. \tag{C.21}$$

This integral representation is nevertheless useful for working out the explicit perturbative expansion of the gauge energy to any desired order.

C.2 The circular string

C.2.1 Semi-classical string solution

The Bethe solution of the circular string makes the ansatz that there is a single contour $\mathcal{C}$ which is purely imaginary and symmetric w.r.t. reflection around the real axis. The (pseudo)density $\sigma(x)$ is assumed to be a constant $\sigma(x) = -2im$, m integer, on the interval $x \in [-ic, ic]$, but non-constant on the intervals $[ic, id]$ and $[-id, -ic]$. It is convenient to rotate the spectral x-plane by $\frac{\pi}{2}$ and redefine $x = iy$. This leads to the classical Bethe equation

$$⨍_c^d dy'\,\frac{i\sigma(iy')\,y}{y^2-y'^2} = \frac{\mathcal{E}}{4}\,\frac{y}{y^2+\frac{1}{2}g^2} - m\log\frac{y+c}{y-c} \tag{C.22}$$

with the normalization

$$2cm\left(1-\frac{g^2}{2c^2}\right)+\int_c^d dy\, i\sigma(iy)\,\left(1+\frac{g^2}{2y^2}\right)=\frac{\alpha}{2}\,, \tag{C.23}$$

and

$$i\mathcal{G}(iy)=2m\log\frac{c-y}{c+y}+2y\int_c^d dy'\,\frac{i\sigma(iy')}{y'^2-y^2}=i\sum_{r=1}^{\infty}(iy)^{2r-1}\mathcal{Q}_{2r}\,, \tag{C.24}$$

where

$$\mathcal{Q}_{2r}=\frac{4}{2r-1}\,\frac{mi}{(ic)^{2r-1}}+2\int_c^d dy\,\frac{i\sigma(iy)}{(iy)^{2r}}\,. \tag{C.25}$$

The expression for the string energy (3.20) is $\mathcal{E}=1+g^2\mathcal{Q}_2(g)$. The solution of (C.22) is, with $q'=c^2/d^2$,

$$\begin{aligned} i\sigma(iy) &= 2m-\frac{4m}{\pi yd}\sqrt{(d^2-y^2)(y^2-c^2)}\,\Pi\left(\frac{y^2}{d^2},q'\right)+\\ &\quad+\frac{\mathcal{E}}{2\pi y}\,\frac{\frac{1}{2}g^2}{y^2+\frac{1}{2}g^2}\sqrt{\frac{(d^2-y^2)(y^2-c^2)}{(d^2+\frac{1}{2}g^2)(c^2+\frac{1}{2}g^2)}}\,, \end{aligned} \tag{C.26}$$

and in a domain near the origin of the y-plane (C.24) is given by

$$\begin{aligned} i\mathcal{G}(iy) &= -\frac{\mathcal{E}}{2}\,\frac{y}{y^2+\frac{1}{2}g^2}+\frac{4m}{yd}\sqrt{(d^2-y^2)(c^2-y^2)}\;\Pi\left(\frac{y^2}{d^2},q'\right)-\\ &\quad-\frac{\mathcal{E}}{2y}\,\frac{\frac{1}{2}g^2}{y^2+\frac{1}{2}g^2}\sqrt{\frac{(d^2-y^2)(c^2-y^2)}{(d^2+\frac{1}{2}g^2)(c^2+\frac{1}{2}g^2)}}\,. \end{aligned} \tag{C.27}$$

The known behavior of the resolvent at $y=0$, namely $\mathcal{G}(iy)=\frac{0}{y}+\mathcal{Q}_2 iy+\mathcal{O}\left(y^3\right)$, yields two conditions:

$$\frac{\mathcal{E}}{\sqrt{(d^2+\frac{1}{2}g^2)(c^2+\frac{1}{2}g^2)}}=\frac{8m}{d}\,\mathrm{K}(q') \qquad \text{and} \qquad \mathrm{K}(r)=\frac{1}{8mc}+\frac{g^2}{2c^2}\left(\mathrm{E}(q')-\mathrm{K}(q')\right), \tag{C.28}$$

while a third condition is obtained from the behavior of the resolvent $\mathcal{G}(iy)$ at infinity $y\to\infty$:

$$\alpha=\frac{1}{2}+4md\left(\mathrm{E}(q')-\mathrm{K}(q')\right)-\frac{2m}{d}\,g^2\,\mathrm{K}(q')\,. \tag{C.29}$$

These three equations determine the three unknowns c, d and $\mathcal{E}$ as a function of the filling fraction α and the coupling constant g. One checks that they indeed reproduce the energy of the circular string as first obtained without Bethe ansatz in [25].

There is a special "algebraic" point at half-filling $\alpha=1/2$ already worked out in [35]. Here the cut extends from c to $d=\infty$. Note that $c=c_0=1/4\pi m$ becomes independent of g! The pseudodensity simplifies to a semi-circle law

$$i\sigma(iy)=2m-\frac{2ym}{y^2+\frac{1}{2}g^2}\sqrt{y^2-c_0^2} \tag{C.30}$$

while the resolvent reduces to

$$i\mathcal{G}(iy) = \frac{2\pi m y}{y^2 + \frac{1}{2}g^2}\left(\sqrt{c_0^2 - y^2} - \sqrt{c_0^2 + \frac{1}{2}g^2}\right), \tag{C.31}$$

and the energy becomes

$$\mathcal{E}(g) = \sqrt{1 + \frac{g^2}{2c_0^2}} = \sqrt{1 + 8\pi^2 m^2 g^2}\,, \tag{C.32}$$

as originally found in [23].

C.2.2 All-loop asymptotic gauge solution

Just as in the string theory computation we assume there to be a *condensate* of Bethe roots on the imaginary axis: $\rho(\varphi) = -2im$, on the interval $[-ic, ic]$, where m is an integer. Outside this interval the root density is again non-constant. Due to the condensate cut it is convenient to perform a rotation $\varphi = i\phi$. Thus the circular string Bethe equation of the last section C.2.1 is replaced in the perturbative gauge theory by the two-cut singular integral equation

$$⨍_c^d \frac{d\phi'\, i\rho(i\phi')\,\phi}{\phi^2 - \phi'^2} = \frac{1}{4}\frac{1}{\sqrt{\phi^2 + 2g^2}} - m\log\frac{\phi + c}{\phi - c} \qquad \text{with} \qquad 2cm + \int_c^d d\phi\, i\rho(i\phi) = \frac{\alpha}{2}\,. \tag{C.33}$$

The ϕ-resolvent is

$$i\bar{\mathbf{G}}(i\phi) = 2m\log\frac{c - \phi}{c + \phi} + 2\phi\int_c^d d\phi'\,\frac{i\rho(i\phi')}{\phi'^2 - \phi^2} = i\sum_{r=1}^{\infty}(i\phi)^{2r-1}\bar{\mathbf{Q}}_{2r}\,, \tag{C.34}$$

with

$$\bar{\mathbf{Q}}_{2r} = \frac{4}{2r-1}\frac{im}{(ic)^{2r-1}} + 2\int_c^d d\phi\,\frac{i\rho(i\phi)}{(i\phi)^{2r}}\,. \tag{C.35}$$

The form of the resolvent appropriate for the expansion in local charges is found to be

$$\begin{aligned} i\bar{\mathbf{G}}(i\phi) &= \frac{4m}{\phi\, d}\sqrt{(d^2 - \phi^2)(c^2 - \phi^2)}\,\Pi\left(\frac{\phi^2}{d^2}, q'\right) - \\ &\quad - \frac{1}{\pi\phi}\frac{d^2}{c\sqrt{d^2 + 2g^2}}\sqrt{\frac{c^2 - \phi^2}{d^2 - \phi^2}}\,\Pi\left(\frac{1 - q'}{q'}\frac{\phi^2}{d^2 - \phi^2}, -\frac{1 - q'}{q'}\frac{2g^2}{d^2 + 2g^2}\right) \end{aligned} \tag{C.36}$$

with $q' = \frac{c^2}{d^2}$. The behavior of the resolvent at $\phi = 0$, namely $\bar{\mathbf{G}}(i\phi) = \frac{0}{\phi} + \mathcal{O}(\phi)$, yields the condition

$$\mathrm{K}(q') = \frac{1}{4\pi m}\frac{d}{c\sqrt{d^2 + 2g^2}}\mathrm{K}\left(\frac{q' - 1}{q'}\frac{2g^2}{d^2 + 2g^2}\right), \tag{C.37}$$

and, after analytic continuation of the representation (C.36) to large values of ϕ, we find from the behavior of $\bar{\mathbf{G}}(i\phi)$ at infinity a second condition

$$\alpha = 4md\,\big(\mathrm{E}(q') - \mathrm{K}(q')\big) + \frac{1}{\pi}\frac{d^2}{c\sqrt{d^2 + 2g^2}}\,\Pi\left(\frac{q' - 1}{q'}, \frac{q' - 1}{q'}\frac{2g^2}{d^2 + 2g^2}\right). \tag{C.38}$$

Finally the density is once more obtained from the behavior of $\bar{\mathbf{G}}(i\phi)$ on the cut. We found the following form

$$i\rho(i\phi) = 2m - \frac{4m}{\pi\phi d}\sqrt{(d^2-\phi^2)(\phi^2-c^2)}\,\Pi\left(\frac{\phi^2}{d^2}, q'\right) + \frac{4dm}{\pi\phi}\sqrt{\frac{\phi^2-c^2}{d^2-\phi^2}}\,\mathrm{K}(q') - \\ -\frac{1}{\pi^2\,\phi}\frac{d^2}{c\sqrt{d^2+2g^2}}\sqrt{\frac{\phi^2-c^2}{d^2-\phi^2}}\,\Pi\left(-\frac{d^2-\phi^2}{d^2+2g^2}\frac{2g^2}{\phi^2}, -\frac{1-q'}{q'}\frac{2g^2}{d^2+2g^2}\right), \quad \text{(C.39)}$$

which should be compared to its much simpler string analog (C.26). This density yields, in view of (3.9), (3.6), integral representations for all proper gauge charges $\mathbf{Q}_{2r}$.

At the half-filling point $\alpha = \frac{1}{2}$ these expressions simplify, but, unlike the string case section C.2.1, they do *not* become algebraic. The density reduces to

$$i\rho(i\phi) = 2m - \frac{1}{\pi^2\,c\,\phi}\sqrt{\phi^2-c^2}\,\Pi\left(-\frac{2g^2}{\phi^2}, -\frac{2g^2}{c^2}\right), \quad \text{(C.40)}$$

and the resolvent becomes

$$i\bar{\mathbf{G}}(i\phi) = \frac{m}{4} - \frac{m}{4}\frac{\phi}{\sqrt{\phi^2+2g^2}} + \frac{1}{2\pi c}\sqrt{c^2-\phi^2}\,\Pi\left(-\frac{2g^2}{\phi^2}, -\frac{2g^2}{c^2}\right), \quad \text{(C.41)}$$

while the boundary point c remains coupling constant dependent at $\alpha = \frac{1}{2}$, in contradistinction to the string theory case:

$$mc = \frac{1}{2\pi^2}\,\mathrm{K}\left(-\frac{2g^2}{c^2}\right). \quad \text{(C.42)}$$

In order to work out the perturbative gauge energy we still have to perform the following integral:

$$\mathbf{D}(g) = \frac{3}{2} - 2\pi mc + \frac{2}{\pi^2 mc}\int_c^\infty d\phi\sqrt{\frac{\phi^2-c^2}{\phi^2+2g^2}} \\ \left(\Pi\left(-\frac{2g^2}{\phi^2}, -\frac{2g^2}{c^2}\right) - \sqrt{1+\frac{2g^2}{\phi^2}}\,\mathrm{K}\left(-\frac{2g^2}{c^2}\right)\right). \quad \text{(C.43)}$$

We did not succeed in calculating this integral in terms of algebraic or elliptic functions. It is however straightforward to use the representation (C.43) to derive the following perturbative expansion of the energy ($c_0 = \frac{1}{4\pi m}$):

$$\mathbf{D}(g) = 1 + \frac{1}{2}\frac{g^2}{2c_0^2} - \frac{1}{8}\left(\frac{g^2}{2c_0^2}\right)^2 + \frac{3}{128}\left(\frac{g^2}{2c_0^2}\right)^4 - \frac{3}{256}\left(\frac{g^2}{2c_0^2}\right)^6 + \frac{267}{32768}\left(\frac{g^2}{2c_0^2}\right)^8 - \\ -\frac{441}{65536}\left(\frac{g^2}{2c_0^2}\right)^{10} + \frac{6483}{1048576}\left(\frac{g^2}{2c_0^2}\right)^{12} - \frac{12813}{2097152}\left(\frac{g^2}{2c_0^2}\right)^{14} + \mathcal{O}\left(g^{16}\right). \quad \text{(C.44)}$$

Oddly, the odd powers of g^2 are missing, except for the linear term. This contrasts with the much simpler string result (C.32), whose expansion matches (C.44) only up to $\mathcal{O}\left(g^4\right)$, and certainly contains all powers of g^2.

D. A density for the string Bethe ansatz

The normalization of the string Bethe equations (3.16) differs from the one in gauge theory (3.10). To make the ansätze more similar we should transform the pseudodensity to a true density.[29] For that purpose we set

$$\sigma(x) = \frac{\rho(x)}{1 - \frac{g^2}{2x^2}} \tag{D.1}$$

and obtain a proper normalization

$$\int_{\mathcal{C}} dx\, \rho(x) = \alpha\,. \tag{D.2}$$

The local charges are now given by

$$\mathcal{Q}_r = \int_{\mathcal{C}} \frac{dx\, \rho(x)}{1 - \frac{g^2}{2x^2}} \frac{1}{x^r}\,. \tag{D.3}$$

Assuming that ρ transforms as a density, $dx\, \rho(x) = d\varphi\, \rho(\varphi)$, we see immediate agreement with the gauge theory expression (3.6), (3.9)

$$\mathbf{Q}_r = \int_{\mathbf{C}} \frac{d\varphi\, \rho(\varphi)}{\sqrt{\varphi^2 - 2g^2}\, x(\varphi)^{r-1}}\,, \tag{D.4}$$

noting a relation which holds by virtue of $x = x(\varphi)$ (2.29), (2.36)

$$x - \frac{g^2}{2x} = \sqrt{\varphi^2 - 2g^2}\,. \tag{D.5}$$

The string Bethe equation using the density reads

$$2 \fint_{\mathcal{C}} \frac{dx'\, \rho_{\mathrm{s}}(x')}{x - x'} = \frac{1}{x} + 2\pi n_\nu \left(1 - \frac{g^2}{2x^2}\right) - \frac{g^2}{x^2} \mathcal{Q}_1\,. \tag{D.6}$$

In contrast, the gauge Bethe equations read

$$2 \fint_{\mathcal{C}} dx'\, \rho_{\mathrm{g}}(x') \left(\frac{1}{x - x'} + \frac{g^2}{2x^2 x'} \frac{1}{1 - \frac{g^2}{2xx'}}\right) = \frac{1}{x} + 2\pi n_\nu \left(1 - \frac{g^2}{2x^2}\right)\,. \tag{D.7}$$

The only distinction between the two is the slightly different second part of the integrand after substituting (D.3) in (D.6).

E. Proof of a curious observation

Let us investigate the leading order perturbative difference between the gauge Bethe equation and the string Bethe equation. In the spectral φ-plane the former is given by (3.12)

[29] This turns out to be the key to a possible generalization of the Bethe ansatz to account for string sigma model quantum effects. These $1/J$ corrections require a clean definition of individual roots.

while the latter is (3.28). Expansion in g gives for the respective equations to three-loop order

$$2 \fint_{\mathbf{C}} \frac{d\varphi' \, \rho(\varphi')}{\varphi - \varphi'} = 2\pi n_\nu + \frac{1}{\varphi} + \frac{g^2}{\varphi^3} + \frac{3g^4}{2\varphi^5} + \mathcal{O}\left(g^6\right) , \tag{E.1}$$

and

$$2 \fint_{\mathbf{C}} \frac{d\varphi' \, \rho_{\mathrm{s}}(\varphi')}{\varphi - \varphi'} = 2\pi n_\nu + \frac{1}{\varphi} - \frac{g^4}{2\varphi^2} \bar{\mathcal{Q}}_{3,0} + \frac{1}{\varphi^3}\left(g^2 + \frac{1}{2} g^4 \, \bar{\mathcal{Q}}_{2,0}\right) + \frac{3g^4}{2\varphi^5} + \mathcal{O}\left(g^6\right) . \tag{E.2}$$

Here we needed also the one-loop second and third moments $\bar{\mathcal{Q}}_{2,0}, \bar{\mathcal{Q}}_{3,0}$ which are obtained from the loop expansion of the string theory φ-moments:

$$\bar{\mathcal{Q}}_r(g) = \sum_{\ell=1}^{\infty} \bar{\mathcal{Q}}_{r,2\ell-2} \, g^{2\ell-2} \qquad \text{with } \bar{\mathcal{Q}}_r = \int_{\mathbf{C}} \frac{d\varphi \, \rho_{\mathrm{s}}(\varphi)}{\varphi^r} . \tag{E.3}$$

To two-loop order the right hand sides of equations (E.1), (E.2) are identical, but for three loops the string equation has two extra terms. The first is proportional to $1/\varphi^2$. As terms even in φ are completely absent in the gauge potential, generic solutions will irreparably differ in structure starting from this order. However, note that this term is multiplied by an odd expectation value $\bar{\mathcal{Q}}_{3,0}$. It is therefore absent for (unpaired) solutions symmetric in φ such as the ones studied in appendix C. Furthermore, we see that for symmetric solutions we can introduce a shifted coupling constant

$$g_{\mathrm{s}}^2 := g^2 + \frac{1}{2} g^4 \, \bar{\mathcal{Q}}_{2,0} , \tag{E.4}$$

and rewrite the string equation (E.2) as

$$2 \fint_{\mathbf{C}} \frac{d\varphi' \, \rho_{\mathrm{s}}(\varphi')}{\varphi - \varphi'} = 2\pi n_\nu + \frac{1}{\varphi} + \frac{g_{\mathrm{s}}^2}{\varphi^3} + \frac{3g_{\mathrm{s}}^4}{2\varphi^5} + \mathcal{O}\left(g_{\mathrm{s}}^6\right) \qquad \text{if } \bar{\mathcal{Q}}_{3,0} = 0 . \tag{E.5}$$

Therefore the equation to be solved is formally identical to the gauge equation (E.1), and, to this order, one will find the same form of the density, but with the shifted coupling (E.4). The string charges can then be obtained from the gauge charges by the simple replacement (E.4). This immediately leads to the result

$$\bar{\mathcal{Q}}_{2r}(g) - \bar{\mathbf{Q}}_{2r}(g) = \frac{1}{2} g^4 \, \bar{\mathcal{Q}}_{2,0} \, \bar{\mathcal{Q}}_{2r,2} + \mathcal{O}\left(g^6\right) , \tag{E.6}$$

which, after accounting for somewhat altered normalizations and conventions, precisely proves in generality the finding in equation (17) in [30], originally derived for two specific solutions (folded and circular string). Likewise, using (3.3), (3.20), we find for *all* even solutions of the Bethe equations

$$\mathcal{E}(g) - \mathbf{D}(g) = \frac{1}{2} g^4 \, \bar{\mathcal{Q}}_{2,0} \, \bar{\mathcal{Q}}_{2,2} + \mathcal{O}\left(g^8\right) , \tag{E.7}$$

which is, after adjusting conventions, the general proof for the "curious observation" at the end of [18].

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